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THE GENERAL SOLUTION AND APPROXIMATIONS OF A DECIC TYPE FUNCTIONAL EQUATION IN VARIOUS NORMED SPACES

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Abstract. In the current work, we define and find the general solution of the decic functional equation

$$\begin{aligned} g(x+5y) &- 10g(x+4y) + 45g(x+3y) - 120g(x+2y) \\ &+ 210g(x+y) - 252g(x) + 210g(x-y) - 120g(x-2y) \\ &+ 45g(x-3y) - 10g(x-4y) + g(x-5y) = 10!g(y) \end{aligned}$$

where $10! = 3628800$. We also investigate and establish the generalized Ulam-Hyers stability of this functional equation in Banach spaces, generalized 2-normed spaces and random normed spaces by using direct and fixed point methods.

1. Introduction

In [64], Ulam proposed the general Ulam stability problem: When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true? In [31], Hyers gave the first affirmative answer to the question of Ulam for additive functional equations on Banach spaces. Hyers result has since then seen many significant generalizations, both in terms of the control condition used to define the concept of approximate solution [4, 26, 43, 52]. On the other hand, Cădariu and Radu noticed that a fixed point alternative method is very important for the solution of the Ulam problem. In other words, they employed this fixed point method to the investigation of the Cauchy functional equation [20] and for the

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quadratic functional equation [19]. The fixed point method was used for the first time by Baker [9] who applied a variant of Banachs fixed point theorem to obtain the Hyers-Ulam stability of a functional equation in a single variable (for more applications of this method, see [5, 6, 13, 14, 15, 16, 42, 66]). During the last seven decades, the stability problems of various functional equations in several spaces such as intuitionistic fuzzy normed spaces, random normed spaces, non-Archimedean fuzzy normed spaces, Banach spaces, orthogonal spaces and many spaces have been broadly investigated by number of mathematicians; for instance, see [7, 10, 11, 18, 21, 24, 25, 27, 28, 35, 39, 40, 49, 50, 53, 54, 55, 69].

One of the most famous functional equations is the additive functional equation

$$f(x+y) = f(x) + f(y). \tag{1.1}$$

In 1821, it was first solved by A. L. Cauchy in the class of continuous real-valued functions. It is often called an additive Cauchy functional equation in honor of Cauchy. The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional equations are powerful tools in almost every field of natural and social sciences. Every solution of the additive functional equation (1.1) is called an additive function.

The second famous functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.2}$$

is said to be *quadratic functional equation* because the quadratic function $f(x) = ax^2$ is a solution of the functional equation (1.2). A function $f : E_1 \to E_2$ between two vector spaces is quadratic if and only if there exists a unique symmetric biadditive function $f(x) = B(x, x)$ for any $x \in E_1$, where B is given by $B(x, y) = \frac{1}{4}[f(x+y) + f(x-y)]$ [33]. The Hyers-Ulam stability of the quadratic functional equation (1.2) was proved by F. Skof [63] in 1983 for the functions $f : E_1 \to E_2$ where E_1 is a normed space and E_2 is a Banach space. In 1984, P.W. Cholewa [22] demonstrated that the result of Skofs theorem is still valid if the relevant domain E_1 is replaced by an Abelian group G. Skof's result was further extended by S. Czerwik [23] to the Hyers-Ulam-Rassias stability of the quadratic functional equation (1.2).

J. M. Rassias [44] introduced the following *cubic functional equation*

$$g(x+2y) + 3g(x) = 3g(x+y) + g(x-y) + 6g(y) \tag{1.3}$$

and investigated its Ulam stability problem. Also, the *quartic functional equation*

$$F(x+2y)+F(x-2y)+6F(x)=4[F(x+y)+F(x-y)+6F(y)] \tag{1.4}$$

was first introduced by J. M. Rassias [45], who solved its Ulam stability problem (for genreral case of (1.4) see [12]).

The general solution and the generalized Hyers-Ulam-Rassias stability of the generalized mixed type of functional equation

$$\begin{aligned} f(x+ay)+f(x-ay) &= a^2\left[f(x+y)+f(x-y)\right]+2\left(1-a^2\right)f(x) \\ &+\frac{\left(a^4-a^2\right)}{12}\left[f\left(2y\right)+f\left(-2y\right)-4f\left(y\right)-4f\left(-y\right)\right] \end{aligned}$$

for fixed integers a with $a \neq 0, \pm 1$ having solution additive, quadratic, cubic and quartic was discussed by K. Ravi et. al., [58]. Its generalized Ulam-Hyers stability in multi-Banach spaces and non-Archimedean normed spaces via fixed point approach was respectively investigated by T.Z. Xu et. al., [65, 67].

The general solution of *Quintic and Sextic functional equations*

$$\begin{aligned} &f(x+3y)-5f(x+2y)+10f(x+y)-10f(x) \\ &\qquad +5f(x-y)-f(x-2y)=120f(y) \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} &f(x+3y)-6f(x+2y)+15f(x+y)-20f(x)+15f(x-y) \\ &\qquad -6f(x-2y)+f(x-3y)=720f(y) \end{aligned} \tag{1.6}$$

was introduced and investigated on the generalized Ulam-Hyers stability in quasi $\beta-$normed spaces via fixed point method by Xu et. al., [66]. Also, they in [68] introduced and discussed the general solution and generalized Ulam-Hyers stability of *Septic and Octic functional equations*

$$\begin{aligned} &f(x+4y)-7f(x+3y)+21f(x+2y)-35f(x+y)+35f(x) \\ &-21f(x-y)+7f(x-2y)-f(x-3y)=5040f(y) \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} &f(x+4y)-8f(x+3y)+28f(x+2y)-56f(x+y)+70f(x) \\ &\qquad -56f(x-y)+28f(x-2y)-8f(x-3y) \\ &\qquad +f(x-4y)=40320f(y) \end{aligned} \tag{1.8}$$

in quasi $\beta-$normed spaces, respectively.

The upcoming *nonic functional equation* was introduced in [46] as follows:

$$\begin{aligned} f(x+5y)-9f(x+4y)+36f(x+3y)-84f(x+2y)\\ +126f(x+y)-126f(x)+84f(x-y)\\ -36f(x-2y)+9f(x-3y)-f(x-4y)=9!f(y) \end{aligned} \tag{1.9}$$

where $9! = 362880$. Recently, the general solution in vector space and generalized Ulam-Hyers stability of the following (1.9) in a Banach Space, Felbin's type fuzzy normed space and Intuitionistic fuzzy normed space by using the standard direct and fixed point methods was introduced and investigated in [17] and [51].

Based on the above investigations, in this paper, we introduce the following Decic functional equation

$$\begin{aligned} g(x+5y)-10g(x+4y)+45g(x+3y)-120g(x+2y)\\ +210g(x+y)-252g(x)+210g(x-y)-120g(x-2y)\\ +45g(x-3y)-10g(x-4y)+g(x-5y)=10!g(y) \end{aligned} \tag{1.10}$$

where $10! = 3628800$. We find the general solution of this new functionl equation. Finally, we study the generalized Ulam-Hyers stability of (1.10) in Banach spaces (BS), generalized 2-normed spaces (G2NS) and random normed space (RNS) using direct and fixed point methods.

2. General solution of (1.10)

In the following result, a solution of the decic functional equation (1.10) is given. For this, let us consider $\mathcal{A}$ and $\mathcal{B}$ be real vector spaces. We notice that the general solution has the generalized polynomial form and it specifies the best mapping g as in Theorem 2.1; see [8].

THEOREM 2.1. *If $g : \mathcal{A} \longrightarrow \mathcal{B}$ be a mapping satisfying* (1.10) *for all $x, y \in \mathcal{A}$, then g is decic.*

Proof. Substituting (x, y) by $(0, 0)$ in (1.10), we see that

$$g(0) = 0. \tag{2.1}$$

Replacing (x, y) by (x, x) in (1.10), we get

$$\begin{aligned} g(6x)-10g(5x)+45g(4x)-120g(3x)+210g(2x)-252g(x)\\ +210g(0)-120g(-x)+45g(-2x)\\ -10g(-3x)+g(-4x)=10!g(x) \end{aligned} \tag{2.2}$$

for all $x \in \mathcal{A}$. Again replacing (x,y) by $(x,-x)$ in (1.10), we obtain

$$\begin{aligned} g(-4x) &- 10g(-3x) + 45g(-2x) - 120g(-x) \\ &+ 210g(0) - 252g(x) + 210g(2x) - 120g(3x) \\ &+ 45g(4x) - 10g(4x) + g(6x) = 10!g(-x) \end{aligned} \tag{2.3}$$

for all $x \in \mathcal{A}$. Subtracting (2.3) from (2.2), we arrive at

$$10!g(x) - 10!g(-x) = 0 \tag{2.4}$$

for all $x \in \mathcal{A}$. It follows from (2.4), we achieve

$$g(-x) = g(x) \tag{2.5}$$

for all $x \in \mathcal{A}$. Hence, g is an even mapping. Setting (x,y) by $(0,2x)$ in (1.10) and using (2.5), we reach

$$2g(10x) - 20g(8x) + 90g(6x) - 240g(4x) - 3628380g(2x) = 0 \tag{2.6}$$

for all $x \in \mathcal{A}$. The above equation can be rewritten as

$$g(10x) - 10g(8x) + 45g(6x) - 120g(4x) - 1814190g(2x) = 0 \tag{2.7}$$

for all $x \in \mathcal{A}$. Again switching (x,y) into $(5x,x)$ in (1.10), we obtain

$$\begin{aligned} g(10x) &- 10g(9x) + 45g(8x) - 120g(7x) \\ &+ 210g(6x) - 252g(5x) + 210g(4x) \\ &- 120g(3x) + 45g(2x) - 3628810g(x) = 0 \end{aligned} \tag{2.8}$$

for all $x \in \mathcal{A}$. Subtracting (2.7) and (2.8), we find

$$\begin{aligned} 10g(9x) &- 55g(8x) + 120g(7x) - 165g(6x) \\ &+ 252g(5x) - 330g(4x) + 120g(3x) \\ &- 1814235g(2x) + 3628810g(x) = 0 \end{aligned} \tag{2.9}$$

for all $x \in \mathcal{A}$. Replacing (x,y) by $(4x,x)$ in (1.10) and using (2.5), we get

$$\begin{aligned} g(9x) &- 10g(8x) + 45g(7x) - 120g(6x) + 210g(5x) \\ &- 252g(4x) + 210g(3x) - 120g(2x) - 3628754g(x) = 0 \end{aligned} \tag{2.10}$$

for all $x \in \mathcal{A}$. Multiplying by 10 on both sides of (2.10), we obtain

$$\begin{aligned} 10g(9x) &- 100g(8x) + 450g(7x) - 1200g(6x) \\ &+ 2100g(5x) - 2520g(4x) + 2100g(3x) \\ &- 1200g(2x) - 36287540g(x) = 0 \end{aligned} \tag{2.11}$$

for all $x \in \mathcal{A}$. Subtracting (2.11) from (2.9), we arrive

$$\begin{aligned} 45g(8x) - 330g(7x) + 1035g(6x) \\ - 1848g(5x) + 2190g(4x) - 1980g(3x) \\ - 1813035g(2x) + 39916350g(x) = 0 \end{aligned} \tag{2.12}$$

for all $x \in \mathcal{A}$. Letting (x, y) by $(3x, x)$ in (1.10) and using (2.5), we have

$$\begin{aligned} g(8x) - 10g(7x) + 45g(6x) - 120g(5x) + 210g(4x) \\ - 252g(3x) + 211g(2x) - 3628930g(x) = 0 \end{aligned} \tag{2.13}$$

for all $x \in \mathcal{A}$. Multiplying by 45 on both sides of (2.13), one obtains

$$\begin{aligned} 45g(8x) - 450g(7x) + 2025g(6x) - 5400g(5x) \\ + 9450g(4x) - 11340g(3x) + 9495g(2x) \\ - 163301850g(x) = 0 \end{aligned} \tag{2.14}$$

for all $x \in \mathcal{A}$. Subtracting equations (2.12) and (2.14), we get

$$\begin{aligned} 120(7x) - 990g(6x) + 3552g(5x) - 7260(4x) \\ + 9360g(3x) - 1822530g(2x) + 203218200g(x) = 0 \end{aligned} \tag{2.15}$$

for all $x \in \mathcal{A}$. Replacing (x, y) by $(2x, x)$ in (1.10) and using (2.5), we obtain

$$\begin{aligned} g(7x) - 10g(6x) + 45g(5x) - 120g(4x) \\ + 211g(3x) - 262g(2x) - 3628545g(x) = 0 \end{aligned} \tag{2.16}$$

for all $x \in \mathcal{A}$. Multiplying by 120 on both sides of (2.16), one finds

$$\begin{aligned} 120g(7x) - 1200g(6x) + 5400g(5x) - 14400g(4x) \\ + 25320g(3x) - 31440g(2x) - 435425400g(x) = 0 \end{aligned} \tag{2.17}$$

for all $x \in \mathcal{A}$. Subtracting equations (2.15) and (2.17), we reach

$$\begin{aligned} 210g(6x) - 1848g(5x) + 7140g(4x) \\ - 15960g(3x) - 1791090g(2x) + 638643600g(x) = 0 \end{aligned} \tag{2.18}$$

for all $x \in \mathcal{A}$. Dividing (2.18) by 2, we arrive at

$$\begin{aligned} 105g(6x) - 924g(5x) + 3570g(4x) \\ - 7980g(3x) - 895545g(2x) + 319321800g(x) = 0 \end{aligned} \tag{2.19}$$

for all $x \in \mathcal{A}$. Interchanging (x, y) into (x, x) in (1.10) and using (2.5), we see that

$$\begin{aligned} &g(6x) - 10g(5x) + 46g(4x) - 130g(3x) \\ &\quad + 255g(2x) - 3629172g(x) = 0 \end{aligned} \tag{2.20}$$

for all $x \in \mathcal{A}$. Multiplying (2.20) by 105, we get

$$\begin{aligned} &105g(6x) - 1050g(5x) + 4830g(4x) - 13650g(3x) \\ &\quad + 26775g(2x) - 381063060g(x) = 0 \end{aligned} \tag{2.21}$$

for all $x \in \mathcal{A}$. Subtracting equations (2.19) and (2.21), we have

$$\begin{aligned} &126g(5x) - 1260g(4x) + 5670g(3x) \\ &\quad - 922320g(2x) + 700384860g(x) = 0 \end{aligned} \tag{2.22}$$

for all $x \in \mathcal{A}$. Replacing (x, y) by $(0, x)$ in (1.10), we obtain

$$2g(5x) - 20g(4x) + 90g(3x) - 240g(2x) - 3628380g(x) = 0 \tag{2.23}$$

for all $x \in \mathcal{A}$. Multiplying (2.23) by 63, one finds that

$$\begin{aligned} &126g(5x) - 1260g(4x) + 5670g(3x) \\ &\quad - 15120g(2x) - 228587940g(x) = 0 \end{aligned} \tag{2.24}$$

for all $x \in \mathcal{A}$. Subtracting equations (2.22) and (2.24), we have

$$-907200g(2x) + 928972800g(x) = 0 \tag{2.25}$$

for all $x \in \mathcal{A}$. It follows from (2.25), we reach

$$g(2x) = 2^{10}g(x) \tag{2.26}$$

for all $x \in \mathcal{A}$. Hence g is a Decic mapping. This completes the proof. □

In the following result which is analogous to Theorem 2.1, we bring another proof for it.

THEOREM 2.2. *If $g : \mathcal{A} \longrightarrow \mathcal{B}$ be a mapping satisfying* (1.10) *for all $x, y \in \mathcal{A}$, then g is decic.*

Proof. Replacing y by $-y$ in (1.10), we get

$$\begin{aligned} &g(x - 5y) - 10g(x - 4y) + 45g(x - 3y) - 120g(x - 2y) \\ &\quad + 210g(x - y) - 256g(x) + 210g(x + y) - 120g(x + 2y) \\ &\quad + 45g(x + 3y) - 10g(x + 4y) + g(x + 5y) = 10!g(-y) \end{aligned} \tag{2.27}$$

for all $x, y \in \mathcal{A}$. Subtracting (2.27) from (1.10) and replacing y by x, we have

$$g(-x) = g(x)$$

for all $x \in \mathcal{A}$. It follows from (2.18) that

$$\begin{aligned} &210g(6x) - 1848g(5x) + 7140g(4x) \\ &\qquad - 15960g(3x) - 1791090g(2x) + 638643600g(x) = 0 \end{aligned} \tag{2.28}$$

for all $x \in \mathcal{A}$. Replacing (x, y) by (x, x) in (1.10) and using (2.5), we find

$$\begin{aligned} &g(6x) - 10g(5x) + 46g(4x) - 130g(3x) \\ &\qquad + 255g(2x) - 3629172g(x) = 0 \end{aligned} \tag{2.29}$$

for all $x \in \mathcal{A}$. Multiplying (2.29) by 210, one finds

$$\begin{aligned} &210g(6x) - 2100g(5x) + 9660g(4x) - 27300g(3x) \\ &\qquad + 53550g(2x) - 762126120g(x) = 0 \end{aligned} \tag{2.30}$$

for all $x \in \mathcal{A}$. Subtracting equations (2.28) and (2.30), we arrive at

$$\begin{aligned} &252g(5x) - 2520g(4x) + 11340g(3x) \\ &\qquad - 1844640g(2x) + 1400769720g(x) = 0 \end{aligned} \tag{2.31}$$

for all $x \in \mathcal{A}$. Replacing (x, y) by $(0, x)$ in (1.10), we obtain

$$\begin{aligned} &2g(5x) - 20g(4x) + 90g(3x) \\ &\qquad - 240g(2x) - 3628380g(x) = 0 \end{aligned} \tag{2.32}$$

for all $x \in \mathcal{A}$. Multiplying (2.32) by 126, one can show that

$$\begin{aligned} &252(5x) - 2520g(4x) + 11340g(3x) \\ &\qquad - 30240g(2x) - 457175880g(x) = 0 \end{aligned} \tag{2.33}$$

for all $x \in \mathcal{A}$. Subtracting equations (2.31) and (2.33), we have

$$-1814400g(2x) + 1857945600g(x) = 0 \tag{2.34}$$

for all $x \in \mathcal{A}$. It follows from (2.34), we reach

$$g(2x) = 2^{10}g(x)$$

for all $x \in \mathcal{A}$. Hence g is a Decic mapping. □

3. Stability results in Banach space

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.10) in Banach space using direct and fixed point methods.

Throughout this section, let us consider $\mathcal{E}$ be a normed space and $\mathcal{F}$ be a Banach space. Define a mapping $Dg_{10} : \mathcal{E} \longrightarrow \mathcal{F}$ by

$$\begin{aligned} Dg_{10}(x,y) =& g(x+5y) - 10g(x+4y) + 45g(x+3y) - 120g(x+2y) \\ &+ 210g(x+y) - 252g(x) + 210g(x-y) - 120g(x-2y) \\ &+ 45g(x-3y) - 10g(x-4y) + g(x-5y) - 10!g(y), \end{aligned}$$

where $10! = 3628800$ for all $x, y \in \mathcal{E}$.

3.1. Banach space: Direct method

THEOREM 3.1. *Let $b = \pm 1$ and $\kappa : \mathcal{E}^2 \longrightarrow [0,\infty)$ be a function such that*

$$\sum_{d=0}^{\infty} \frac{\kappa\left(2^{bd}x, 2^{bd}y\right)}{2^{10bd}}, \text{converges in } \mathbb{R} \text{ and } \lim_{d\to\infty} \frac{\kappa\left(2^{bd}x, 2^{bd}y\right)}{2^{10bd}} = 0 \tag{3.1}$$

for all $x, y \in \mathcal{E}$. Let $Dg_{10} : \mathcal{E} \longrightarrow \mathcal{F}$ be a mapping fulfilling the inequality

$$\|Dg_{10}(x,y)\| \le \kappa(x,y) \tag{3.2}$$

for all $x, y \in \mathcal{E}$. Then, there exists a unique decic mapping $\mathcal{T} : \mathcal{E} \longrightarrow \mathcal{F}$ which satisfies (1.10) *and*

$$\|g(x) - \mathcal{T}(x)\| \le \frac{1}{2^{10}} \sum_{d=\frac{1-b}{2}}^{\infty} \frac{K\left(2^{bd}x, 2^{bd}x\right)}{2^{10bd}} \tag{3.3}$$

where $K\left(2^{bd}x, 2^{bd}x\right)$ and $\mathcal{T}(x)$ are defined by

$$\begin{aligned} K\left(2^{bd}x, 2^{bd}x\right) = & \frac{1}{907200}\left\{\frac{1}{2}\left[\frac{1}{2}\cdot\kappa(0, 2\cdot 2^{bd}x) + \kappa(5\cdot 2^{bd}x, 2^{bd}x)\right.\right. \\ &\left. + 45\kappa(3\cdot 2^{bd}x, 2^{bd}x) + 120\kappa(2\cdot 2^{bd}x, 2^{bd}x)\right] \\ &\left. + 10\kappa(4\cdot 2^{bd}x, 2^{bd}x) + 105\kappa(2^{bd}x, 2^{bd}x) + 63\kappa(0, 2^{bd}x)\right\} \end{aligned} \tag{3.4}$$

and

$$\mathcal{T}(x) = \lim_{d\to\infty} \frac{g(2^{bd}x)}{2^{10bd}} \tag{3.5}$$

for all $x \in \mathcal{E}$, respectively.

Proof. Changing (x,y) by $(0,2x)$ in (3.2), we get

$$\begin{aligned} &\left\|2g(10x) - 20g(8x) + 90g(6x) - 240g(4x) - 3628380g(2x)\right\| \\ &\qquad \le \kappa(0,2x) \end{aligned} \tag{3.6}$$

for all $x \in \mathcal{E}$. Dividing the above inequality by 2, one can arrive

$$\begin{aligned} &\Big\| g(10x) - 10g(8x) + 45g(6x) - 120g(4x) - 1814190g(2x) \Big\| \\ &\qquad \le \frac{1}{2}\kappa(0, 2x) \end{aligned} \tag{3.7}$$

for all $x \in \mathcal{E}$. Again setting (x, y) by $(5x, x)$ in (3.2), we obtain

$$\begin{aligned} &\Big\| g(10x) - 10g(9x) + 45g(8x) - 120g(7x) + 210g(6x) - 252g(5x) \\ &\qquad + 210g(4x) - 120g(3x) + 45g(2x) - 3628810g(x) \Big\| \\ &\qquad \le \kappa(5x, x) \end{aligned} \tag{3.8}$$

for all $x \in \mathcal{E}$. Combining (3.7) and (3.8), we have

$$\begin{aligned} &\Big\| 10g(9x) - 55g(8x) + 120g(7x) - 165g(6x) + 252g(5x) \\ &\quad - 330g(4x) + 120g(3x) - 1814235g(2x) + 3628810g(x) \Big\| \\ &= \Big\| g(10x) - 10g(8x) + 45g(6x) - 120g(4x) - 1814190g(2x) \\ &\quad - g(10x) + 10g(9x) - 45g(8x) + 120g(7x) - 210g(6x) + 252g(5x) \\ &\quad - 210g(4x) + 120g(3x) - 45g(2x) + 3628810g(x) \Big\| \\ &\le \Big\| g(10x) - 10g(8x) + 45g(6x) - 120g(4x) - 1814190g(2x) \Big\| \\ &\quad + \Big\| g(10x) - 10g(9x) + 45g(8x) - 120g(7x) + 210g(6x) - 252g(5x) \\ &\quad + 210g(4x) - 120g(3x) + 45g(2x) - 3628810g(x) \Big\| \\ &\le \frac{1}{2}\kappa(0, 2x) + \kappa(5x, x) \end{aligned} \tag{3.9}$$

for all $x \in \mathcal{E}$. Switching (x, y) into $(4x, x)$ in (3.2) and using evenness of g, we get

$$\begin{aligned} &\Big\| g(9x) - 10g(8x) + 45g(7x) - 120g(6x) + 210g(5x) - 252g(4x) \\ &\qquad + 210g(3x) - 120g(2x) - 3628754g(x) \Big\| \\ &\qquad \le \kappa(4x, x) \end{aligned} \tag{3.10}$$

for all $x \in \mathcal{E}$. Multiplying by 10 on both sides of (3.10), we see that

$$\begin{aligned}\Big\|10g(9x) - 100g(8x) + 450g(7x) - 1200g(6x) + 2100g(5x)& \\ - 2520g(4x) + 2100g(3x) - 1200g(2x) - 36287540g(x)\Big\|& \\ \le 10\kappa(4x, x)& \qquad (3.11)\end{aligned}$$

for all $x \in \mathcal{E}$. It follows from (3.9) and (3.11) that

$$\begin{aligned}\Big\|45g(8x) - 330g(7x) + 1035g(6x) - 1848g(5x) + 2190g(4x) - 1980g(3x)& \\ - 1813035g(2x) + 39916350g(x)\Big\|& \\ \le \frac{1}{2} \cdot \kappa(0, 2x) + \kappa(5x, x) + 10\kappa(4x, x)& \qquad (3.12)\end{aligned}$$

for all $x \in \mathcal{E}$. Letting (x, y) by $(3x, x)$ in (3.2) and applying evenness of g, we have

$$\begin{aligned}\Big\|g(8x) - 10g(7x) + 45g(6x) - 120g(5x) + 210g(4x)& \\ - 252g(3x) + 211g(2x) - 3628930g(x)\Big\| \le \kappa(3x, x)& \qquad (3.13)\end{aligned}$$

for all $x \in \mathcal{E}$. Multiplying by 45 on both sides of (3.13), one obtains

$$\begin{aligned}\Big\|45g(8x) - 450g(7x) + 2025g(6x) - 5400g(5x) + 9450g(4x)& \\ - 11340g(3x) + 9495g(2x) - 163301850g(x)\Big\| \le 45\kappa(3x, x)& \qquad (3.14)\end{aligned}$$

for all $x \in \mathcal{E}$. The relations (3.12) and (3.14) imply that

$$\begin{aligned}\Big\|120(7x) - 990g(6x) + 3552g(5x) - 7260(4x) + 9360g(3x)& \\ - 1822530g(2x) + 203218200g(x)\Big\|& \\ \le \frac{1}{2}\kappa(0, 2x) + \kappa(5x, x) + 10\kappa(4x, x) + 45\kappa(3x, x)& \qquad (3.15)\end{aligned}$$

for all $x \in \mathcal{E}$. Replacing (x, y) by $(2x, x)$ in (3.2) and using evenness of g, we obtain

$$\begin{aligned}\Big\|g(7x) - 10g(6x) + 45g(5x) - 120g(4x)& \\ + 211g(3x) - 262g(2x) - 3628545g(x)\Big\| \le \kappa(2x, x)& \qquad (3.16)\end{aligned}$$

for all $x \in \mathcal{E}$. Multiplying by 120 on both sides of (3.16), one finds

$$\begin{aligned} &\Big\| 120g(7x) - 1200g(6x) + 5400g(5x) - 14400g(4x) \\ &+ 25320g(3x) - 31440g(2x) - 435425400g(x) \Big\| \leq 120\kappa(2x, x) \end{aligned} \tag{3.17}$$

for all $x \in \mathcal{E}$. It follows from (3.15) and (3.17) that

$$\begin{aligned} &\Big\| 210g(6x) - 1848g(5x) + 7140g(4x) \\ &- 15960g(3x) - 1791090g(2x) + 638643600g(x) \Big\| \leq \Gamma(x) \end{aligned} \tag{3.18}$$

for all $x \in \mathcal{E}$, where

$$\Gamma(x) = \left[\frac{1}{2}\kappa(0, 2x) + \kappa(5x, x) + 10\kappa(4x, x) + 45\kappa(3x, x) + 120\kappa(2x, x) \right].$$

Dividing (3.18) by 2, we arrive at

$$\begin{aligned} &\Big\| 105g(6x) - 924g(5x) + 3570g(4x) \\ &- 7980g(3x) - 895545g(2x) + 319321800g(x) \Big\| \leq \frac{1}{2}\Gamma(x) \end{aligned} \tag{3.19}$$

for all $x \in \mathcal{E}$. Substituting (x, y) by (x, x) in (3.2) and using evenness of g, we get

$$\begin{aligned} &\Big\| g(6x) - 10g(5x) + 46g(4x) - 130g(3x) + 255g(2x) - 3629172g(x) \Big\| \\ &\leq \kappa(x, x) \end{aligned} \tag{3.20}$$

for all $x \in \mathcal{E}$. Multiplying (3.20) by 105, we have

$$\begin{aligned} &\Big\| 105g(6x) - 1050g(5x) + 4830g(4x) - 13650g(3x) \\ &+ 26775g(2x) - 381063060g(x) \Big\| \leq 105\kappa(x, x) \end{aligned} \tag{3.21}$$

for all $x \in \mathcal{E}$. It follows from (3.19) and (3.21) that

$$\begin{aligned} &\Big\| 126g(5x) - 1260g(4x) + 5670g(3x) - 922320g(2x) + 700384860g(x) \Big\| \\ &\leq \frac{1}{2}\Gamma(x) + 105\kappa(x, x) \end{aligned} \tag{3.22}$$

for all $x \in \mathcal{E}$. Replacing (x, y) by $(0, x)$ in (3.2) and using evenness of g, we obtain

$$\begin{aligned} &\Big\| 2g(5x) - 20g(4x) + 90g(3x) - 240g(2x) - 3628380g(x) \Big\| \\ &\leq \kappa(0, x) \end{aligned} \tag{3.23}$$

for all $x \in \mathcal{E}$. Multiplying (3.23) by 63, we obtain

$$\begin{aligned}&\Big\|126g(5x) - 1260g(4x) + 5670g(3x) - 15120g(2x) - 228587940g(x)\Big\| \\ &\qquad \le 63\kappa(0,x) \end{aligned} \tag{3.24}$$

for all $x \in \mathcal{E}$. Combining (3.22) and (3.24), we get

$$\begin{aligned}&\Big\| - 907200g(2x) + 928972800g(x)\Big\| \\ &\qquad \le \frac{1}{2}\Gamma(x) + 105\kappa(x,x) + 63\kappa(0,x) \end{aligned} \tag{3.25}$$

for all $x \in \mathcal{E}$. The relation (3.25) implies that

$$\begin{aligned}&\Big\|g(2x) - 1024g(x)\Big\| \\ &\qquad \le \frac{1}{907200}\frac{\Gamma(x)}{2} + 105\kappa(x,x) + 63\kappa(0,x) \end{aligned} \tag{3.26}$$

for all $x \in \mathcal{E}$. Define

$$K(x,x) = \frac{1}{907200}\frac{\Gamma(x)}{2} + 105\kappa(x,x) + 63\kappa(0,x) \tag{3.27}$$

for all $x \in \mathcal{E}$. From (3.27), we have

$$\Big\|g(2x) - 2^{10}g(x)\Big\| \le K(x,x) \tag{3.28}$$

for all $x \in \mathcal{E}$. It follows from (3.28) that

$$\Big\|\frac{g(2x)}{2^{10}} - g(x)\Big\| \le \frac{K(x,x)}{2^{10}} \tag{3.29}$$

for all $x \in \mathcal{E}$. Now, by replacing x by $2x$ and dividing by 2^{10} in (3.29), we have

$$\Big\|\frac{g(2^2x)}{2^{20}} - \frac{g(2x)}{2^{10}}\Big\| \le \frac{K(2x,2x)}{2^{20}} \tag{3.30}$$

for all $x \in \mathcal{E}$. From (3.29) and (3.30), we obtain

$$\begin{aligned}\Big\|\frac{g(2^2x)}{2^{20}} - g(x)\Big\| &\le \Big\|\frac{g(2^2x)}{2^{20}} - \frac{g(2x)}{2^{10}}\Big\| + \Big\|\frac{g(2x)}{2^{10}} - g(x)\Big\| \\ &\le \frac{1}{2^{10}}\left[K(x,x) + \frac{K(2x,2x)}{2^{10}}\right] \end{aligned} \tag{3.31}$$

for all $x \in \mathcal{E}$. Generalizing, for a positive integer a, we land

$$\Big\|\frac{g(2^ax)}{2^{10a}} - g(x)\Big\| \le \frac{1}{2^{10}}\sum_{d=0}^{a-1}\frac{K(2^dx,2^dx)}{2^{10d}} \tag{3.32}$$

for all $x \in \mathcal{E}$. To prove the convergence of the sequence $\left\{\frac{g(2^a x)}{2^{10a}}\right\}$, we replace x by $2^d x$ in (3.32) and divide the resultant by 2^{10d}, for any $a, d > 0$. Then, we get

$$\begin{aligned}
\left\|\frac{g(2^{a+d}x)}{2^{10(a+d)}} - \frac{g(2^d x)}{2^{10d}}\right\| &= \frac{1}{2^{10d}}\left\|\frac{g(2^a \cdot 2^d x)}{2^{10a}} - g(2^d x)\right\| \\
&\leq \frac{1}{2^{10d}}\frac{1}{2^{10}}\sum_{c=0}^{a-1}\frac{K(2^c \cdot 2^d x, 2^c \cdot 2^d x)}{2^{10c}} \\
&\leq \frac{1}{2^{10}}\sum_{c=0}^{\infty}\frac{K(2^{c+d}x, 2^{c+d}x)}{2^{10(c+d)}} \\
&\to 0 \quad as \ d \to \infty
\end{aligned}$$

for all $x \in \mathcal{E}$. Thus, it follows that the sequence $\left\{\frac{g(2^a x)}{2^{10a}}\right\}$ is Cauchy in $\mathcal{F}$ and so it converges. Therefore, we see that a mapping $\mathcal{T}(x) : \mathcal{E} \to \mathcal{F}$ defined by

$$\mathcal{T}(x) = \lim_{a\to\infty} \frac{g(2^a x)}{2^{10a}}$$

is well defined for all $x \in \mathcal{E}$. In order to show that $\mathcal{T}$ satisfies (1.10), by interchanging (x, y) into $(2^a x, 2^a y)$ in (3.2) and then dividing by 2^{10a}, we have

$$\|\mathcal{T}(x, y)\| = \lim_{a\to\infty} \frac{1}{2^{10a}} \|Dg_{10}(2^a x, 2^a y)\| \leq \lim_{a\to\infty} \frac{1}{2^{10a}} \kappa(2^a x, 2^a y)$$

for all $x, y \in \mathcal{E}$ and so the mapping $\mathcal{T}$ is decic. Taking the limit as a approaches to infinity in (3.32), we find that the mapping $\mathcal{T}$ is a decic mapping satisfying the inequality (3.3) near the approximate mapping $g : \mathcal{E} \to \mathcal{F}$ of equation (1.10). Hence, $\mathcal{T}$ satisfies (1.10), for all $x, y \in \mathcal{E}$.

To prove that $\mathcal{T}$ is unique, we assume now that there is $\mathcal{T}'$ as another decic mapping satisfying (1.10) and the inequality (3.3). Then, it follows easily that

$$\mathcal{T}(2^a x) = 2^{10a}\mathcal{T}(x), \qquad \mathcal{T}'(2^a x) = 2^{10a}\mathcal{T}'(x)$$

for all $x \in \mathcal{E}$ and all $a \in \mathbb{N}$. Thus

$$\left\|\mathcal{T}(x) - \mathcal{T}'(x)\right\| = \frac{1}{2^{10a}} \left\|\mathcal{T}(2^a x) - \mathcal{T}'(2^a x)\right\|$$

$$\leq \frac{1}{2^{10a}} \left\{ \|\mathcal{T}(2^a x) - g(2^a x)\| + \left\| g(2^a x) - \mathcal{T}'(2^a x) \right\| \right\}$$
$$\leq \frac{1}{2^{10}} \sum_{d=0}^{\infty} \frac{\kappa(2^{c+a}x, 2^{c+a}x)}{2^{10(c+a)}}$$

for all $x \in \mathcal{E}$. Therefore, as $a \to \infty$ in the above inequality, one establishes

$$\mathcal{T}(x) = \mathcal{T}'(x)$$

for all $x \in \mathcal{E}$. Hence, the assertion holds for $b = 1$.
Case (ii): Assume $b = -1$.

Replacing x by $\frac{x}{2}$ in (3.28), we get

$$\left\| g(x) - 2^{10} g\left(\frac{x}{2}\right) \right\| \leq K\left(\frac{x}{2}, \frac{x}{2}\right) \tag{3.33}$$

for all $x \in \mathcal{E}$. The rest of the proof is similar to that of case $b = 1$. Thus, for $b = -1$ the assertion holds as well. This finishes the proof. □

The following corollaries are immediate consequence of Theorem 3.1 concerning the stability of (1.10).

Corollary 3.2. *Let $Dg_{10} : \mathcal{E} \longrightarrow \mathcal{F}$ be a mapping. If there exist real numbers θ and λ such that*

$$\|Dg_{10}(x,y)\| \leq \begin{cases} \theta, & \\ \theta\left\{||x||^{\lambda} + ||y||^{\lambda}\right\}, & \lambda \neq 10; \\ \theta||x||^{\lambda}||y||^{\lambda}, & \lambda \neq 5; \\ \theta\left\{||x||^{\lambda}||y||^{\lambda} + \left\{||x||^{2\lambda} + ||y||^{2\lambda}\right\}\right\}, & \lambda \neq 5; \end{cases}$$

for all $x, y \in \mathcal{E}$, then there exists a unique decic mapping $\mathcal{T} : \mathcal{E} \longrightarrow \mathcal{F}$ such that

$$\|g(x) - \mathcal{T}(x)\| \leq \begin{cases} \dfrac{\theta_C}{|2^{10}-1|}, & \\ \dfrac{\theta_S||x||^{s}}{|2^{10}-2^{\lambda}|}, & \lambda \neq 10, \\ \dfrac{\theta_P||x||^{2\lambda}}{|2^{10}-2^{2\lambda}|}, & \lambda \neq 5, \\ \dfrac{\theta_{SP}||x||^{2\lambda}}{|2^{10}-2^{2\lambda}|}, & \lambda \neq 5, \end{cases}$$

where

$$\theta_C = \frac{1025\ \theta}{3628800},$$
$$\theta_S = \frac{\theta\Big[2\cdot 5^{\lambda}+20\cdot 4^{\lambda}+90\cdot 3^{\lambda}+241\cdot 2^{\lambda}+1444\Big]}{3628800},$$
$$\theta_P = \frac{\theta\Big[5^{\lambda}+10\cdot 4^{\lambda}+45\cdot 3^{\lambda}+120\cdot 2^{\lambda}+210\Big]}{1814400},$$
$$\theta_{SP} = \frac{\theta}{3628800}\Big[2\cdot 5^{2\lambda}+20\cdot 4^{2\lambda}+90\cdot 3^{2\lambda}+241\cdot 2^{2\lambda}+1444\Big] + \frac{\theta}{1814400}\Big[5^{\lambda}+10\cdot 4^{\lambda}+45\cdot 3^{\lambda}+120\cdot 2^{\lambda}+210\Big],$$

for all $x \in \mathcal{E}$.

COROLLARY 3.3. *Let* $Dg_{10} : \mathcal{E} \longrightarrow \mathcal{F}$ *be a mapping. If there exist real numbers* $\theta, \lambda_1, \lambda_2, \lambda = \lambda_1 + \lambda_2$ *such that*

$$\|Dg_{10}(x,y)\| \le \begin{cases} \theta\left\{||x||^{\lambda_1} + ||y||^{\lambda_2}\right\}, & \lambda_1, \lambda_2 \neq 10; \\ \theta||x||^{\lambda_1}||y||^{\lambda_2}, & \lambda \neq 10; \\ \theta\left\{||x||^{\lambda_1}||y||^{\lambda_2} + \left\{||x||^{\lambda} + ||y||^{\lambda}\right\}\right\}, & \lambda \neq 10; \end{cases}$$

for all $x, y \in \mathcal{E}$, *then there exists a unique decic mapping* $\mathcal{T} : \mathcal{E} \longrightarrow \mathcal{F}$ *such that*

$$\|g(x) - \mathcal{T}(x)\| \le \begin{cases} \theta_s, & \lambda_1, \lambda_2 \neq 10, \\ \theta_p, & \lambda \neq 10, \\ \theta_{sp}, & \lambda \neq 10, \end{cases}$$

where

$$\theta_s = \frac{1}{1814400}\left\{\frac{\theta||x||^{\lambda_1}\Big[5^{\lambda_1}+10\cdot 4^{\lambda_1}+45\cdot 3^{\lambda_1}+120\cdot 2^{\lambda_1}+210\Big]}{|2^{10}-2^{\lambda_1}|} + \frac{\theta||x||^{\lambda_2}\Big[2^{\lambda_2}+688\Big]}{|2^{10}-2^{\lambda_2}|}\right\},$$
$$\theta_p = \frac{\theta||x||^{\lambda}}{1814400}\left\{\frac{\Big[5^{\lambda_1}+10\cdot 4^{\lambda_1}+45\cdot 3^{\lambda_1}+120\cdot 2^{\lambda_1}+210\Big]}{|2^{10}-2^{\lambda}|}\right\},$$
$$\theta_{sp} = \frac{\theta||x||^{\lambda}}{1814400\ |2^{10}-2^{\lambda}|}\left\{\Big[5^{\lambda}+10\cdot 4^{\lambda}+45\cdot 3^{\lambda}+120\cdot 2^{\lambda}+210\Big] + \Big[2^{\lambda}+688\Big] + \Big[5^{\lambda_1}+10\cdot 4^{\lambda_1}+45\cdot 3^{\lambda_1}+120\cdot 2^{\lambda_1}+210\Big]\right\},$$

for all $x \in \mathcal{E}$.

3.2. Banach space: Fixed point method

Here, we recall a fundamental result in fixed point theory as follows.

Theorem 3.4. [36] (*The alternative fixed point Theorem*) *Suppose that for a complete generalized metric space* (X, d) *and a strictly contractive mapping* $T : X \to X$ *with Lipschitz constant* L. *Then, for each given element* $x \in X$, *either*
$d(T^n x, T^{n+1} x) = \infty \quad \forall\, n \geq 0$, *or*
there exists a natural number n_0 *such that:*

(FP1) $d(T^n x, T^{n+1} x) < \infty$ *for all* $n \geq n_0$;
(FP2) *The sequence* $(T^n x)$ *is convergent to a fixed point* y^* *of* T;
(FP3) y^* *is the unique fixed point of* T *in the set* $Y = \{y \in X : d(T^{n_0} x, y) < \infty\}$;
(FP4) $d(y^*, y) \leq \frac{1}{1-L}\, d(y, Ty)$ *for all* $y \in Y$.

Using Theorem 3.4, we obtain the generalized Ulam-Hyers stability of (1.10).

Theorem 3.5. *Let* $Dg_{10} : \mathcal{E} \longrightarrow \mathcal{F}$ *be a mapping for which there exists a function* $\zeta : \mathcal{E}^2 \longrightarrow [0, \infty)$ *with the condition*

$$\lim_{n\to\infty} \frac{1}{\wp_i^{10n}} \zeta(\wp_i^n x, \wp_i^n y) = 0 \tag{3.34}$$

where

$$\wp_i = \begin{cases} 2 & if \quad i = 0, \\ \frac{1}{2} & if \quad i = 1 \end{cases} \tag{3.35}$$

such that the functional inequality

$$\|Dg_{10}(x, y)\| \leq \zeta(x, y) \tag{3.36}$$

holds for all $x, y \in \mathcal{E}$. *Assume that there exists* $L = L(i)$ *such that the mapping*

$$D(x, x) = K\left(\frac{x}{2}, \frac{x}{2}\right)$$

where $K(x, x)$ *is defined in* (3.27) *with the property*

$$\frac{1}{\wp_i^{10}} D(\wp_i x, \wp_i x) = L\ D(x, x) \tag{3.37}$$

for all $x \in \mathcal{E}$. *Then, there exists a unique Decic mapping* $\mathcal{T} : \mathcal{E} \longrightarrow \mathcal{F}$ *satisfying the functional equation* (1.10) *and*

$$\| g(x) - \mathcal{T}(x) \| \leq \left(\frac{L^{1-i}}{1-L}\right) D(x, x) \tag{3.38}$$

for all $x \in \mathcal{E}$.

Proof. Consider the set

$$\mathcal{A} = \{h | h : \mathcal{E} \longrightarrow \mathcal{F},\ h(0) = 0\}$$

and introduce the generalized metric $d : \mathcal{A} \times \mathcal{A} \longrightarrow [0, \infty]$ as follows:

$$d(g, h) = \inf\{\omega \in (0, \infty) : \|g(x) - h(x)\| \leq \omega D(x, x), x \in \mathcal{E}\}. \tag{3.39}$$

It is easy to see that $(\mathcal{A}, d)$ is complete with respect to the defined metric. Let us define the linear mapping $J : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$Jh(x) = \frac{1}{\wp_i^{10}} h(\wp_i x),$$

for all $x \in \mathcal{E}$. For given $h, g \in \mathcal{A}$, let $\omega \in [0, \infty)$ be an arbitrary constant with $d(g, h) \leq \omega$, that is

$$\| h(x) - g(x) \| \leq \omega D(x, x), \qquad x \in \mathcal{E}.$$

So, we have

$$\begin{aligned}
\| Jh(x) - Jg(x) \|_{\mathcal{F}} &= \left\| \frac{1}{\wp_i^{10}} h(\wp_i x) - \frac{1}{\wp_i^{10}} g(\wp_i x) \right\| \\
&\leq \frac{\omega}{\wp_i^{10}} D(\wp_i x, \wp_i x) \\
&\leq L\omega D(x, x)
\end{aligned}$$

for all $x \in \mathcal{E}$, that is $d(Jg, Jh) \leq Ld(g, h)$ for all $g, h \in \mathcal{A}$. This implies that J is a strictly contractive mapping on $\mathcal{A}$ with Lipschitz constant L. From (3.28), (3.37) and (3.39) for the case $i = 0$, we get

$$\left\| g(2x) - 2^{10} g(x) \right\| \leq K(x, x), \qquad (x \in \mathcal{E}),$$

and

$$\left\| \frac{g(2x)}{2^{10}} - g(x) \right\| \leq \frac{1}{2^{10}} K(x, x), \qquad (x \in \mathcal{E}).$$

So, we obtain

$$\|Jg(x) - g(x)\| \leq L\ D(x, x), \qquad (x \in \mathcal{E}).$$

Hence,

$$d(Jg, g) \leq L^{1-0} \qquad (g \in \mathcal{A}) \tag{3.40}$$

Replacing $x = \frac{x}{2}$ in (3.37) and (3.40) for the case $i = 1$, we find

$$\left\| g(x) - 2^{10} g\left(\frac{x}{2}\right) \right\| \leq K\left(\frac{x}{2}, \frac{x}{2}\right), \qquad (x \in \mathcal{E}).$$

Then,

$$\|g(x) - Jg(x)\| \leq D(x, x), \qquad (x \in \mathcal{E}),$$

and

$$\|g(x) - Jg(x)\| \leq L^{1-1} D(x,x), \qquad (x \in \mathcal{E}).$$

Thus, we obtain

$$d(Jg, g) \leq L^{1-1} \qquad (g \in \mathcal{A}). \tag{3.41}$$

Therefore, from (3.40) and (3.41), we arrive

$$d(Jg, g) \leq L^{1-i} \qquad (g \in \mathcal{A}).$$

where $i = 0, 1$. Hence, the property (FP1) holds. It follows from property (FP2) that there exists a fixed point $\mathcal{T}$ of J in $\mathcal{A}$ such that

$$\mathcal{T}(x) = \lim_{n\to\infty} \frac{1}{\wp_i^{10n}} g(\wp_i^n x) \tag{3.42}$$

for all $x \in \mathcal{E}$. In order to show that $\mathcal{T}$ satisfies (1.10), replace (x, y) by $(\wp_i^n x, \wp_i^n y)$ in (3.36) and divide by $\wp_i^{10n}$, we have

$$\|\mathcal{T}_{10}(x,y)\| = \lim_{n\to\infty} \frac{1}{\wp_i^{10n}} \|Dg_{10}(\wp_i^n x, \wp_i^n y)\| \leq \lim_{n\to\infty} \frac{1}{\wp_i^{10n}} \zeta\left(\wp_i^n x, \wp_i^n y\right) = 0$$

for all $x, y \in \mathcal{E}$, and so the mapping $\mathcal{T}$ is decic. By property (FP3), $\mathcal{T}$ is the unique fixed point of J in the set

$$\Delta = \{\mathcal{T} \in \mathcal{A} : d(f, \mathcal{T}) < \infty\},$$

Finally, by property (FP4), we obtain

$$\|g(x) - \mathcal{T}(x)\| \leq \frac{L^{1-i}}{1-L}$$

So, the proof is completed. □

Using Theorem 3.5, we prove the following corollary concerning the stability of (1.10).

Corollary 3.6. *Let $Dg_{10} : \mathcal{E} \longrightarrow \mathcal{F}$ be a mapping. If there exist real numbers Θ and ρ such that*

$$\|Dg_{10}(x,y)\| \leq \begin{cases} \Theta, & \\ \Theta\left\{||x||^\rho + ||y||^\rho\right\} & \rho \neq 10; \\ \Theta ||x||^\rho ||y||^\rho & \rho \neq 5; \\ \Theta\left\{||x||^\rho ||y||^\rho + \left\{||x||^{2\rho} + ||y||^{2\rho}\right\}\right\} & \rho \neq 5; \end{cases}$$

for all $x, y \in \mathcal{E}$, then there exists a unique decic mapping $\mathcal{T} : \mathcal{E} \to \mathcal{F}$ such that

$$\|g(x) - \mathcal{T}(x)\| \leq \begin{cases} \dfrac{\Theta_1}{|2^{10} - 1|}, & \\ \dfrac{\Theta_2}{|2^{10} - 2^{\rho}|}, & \rho \neq 10, \\ \dfrac{\Theta_3}{|2^{10} - 2^{2\rho}|}, & \rho \neq 5, \\ \dfrac{\Theta_4}{|2^{10} - 2^{2\rho}|} & \rho \neq 5, \end{cases}$$

where

$$\begin{aligned} \Theta_1 &= \frac{1025\Theta}{3628800}, \\ \Theta_2 &= \frac{\Theta||x||^{\rho}}{362880 \cdot 2^{\rho}} \left[2 \cdot 5^{\rho} + 20 \cdot 4^{\rho} + 90 \cdot 3^{\rho} + 241 \cdot 2^{\rho} + 1444\right] \\ \Theta_3 &= \frac{\Theta||x||^{2\rho}}{1814400 \cdot 2^{2\rho}} \left[5^{\rho} + 10 \cdot 4^{\rho} + 45 \cdot 3^{\rho} + 120 \cdot 2^{\rho} + 210\right] \\ \Theta_4 &= \frac{\Theta||x||^{\rho}}{2^{\rho}} \left\{ \frac{1}{3628800} \left[2 \cdot 5^{2\rho} + 20 \cdot 4^{2\rho} + 90 \cdot 3^{2\rho} + 241 \cdot 2^{2\rho} + 1444\right] \right. \\ &\qquad \left. + \frac{1}{1814400} \left[5^{\rho} + 10 \cdot 4^{\rho} + 45 \cdot 3^{\rho} + 120 \cdot 2^{\rho} + 210\right] \right\} \end{aligned}$$

for all $x \in \mathcal{E}$.

Proof. Let

$$\zeta(x, y) = \begin{cases} \Theta, \\ \Theta \left\{||x||^{\rho} + ||y||^{\rho}\right\} \\ \Theta||x||^{\rho}||y||^{\rho} \\ \Theta \left\{||x||^{\rho}||y||^{\rho} + \left\{||x||^{2\rho} + ||y||^{2\rho}\right\}\right\} \end{cases}$$

for all $x, y \in \mathcal{E}$. Now

$$\frac{1}{\wp_i^{10n}} \zeta(\wp_i^n x, \wp_i^n y) = \begin{cases} \dfrac{\Theta}{\wp_i^{10n}}, \\ \dfrac{\Theta}{\wp_i^{10n}} \{||\wp_i^n x||^{\rho} + ||\wp_i^n y||^{\rho}\}, \\ \dfrac{\Theta}{\wp_i^{10n}} ||\wp_i^n x||^{\rho} \; ||\wp_i^n y||^{\rho} \\ \dfrac{\Theta}{\wp_i^{10n}} \left\{ ||\wp_i^n x||^{\rho} \; ||\wp_i^n y||^{\rho} + \left\{||\wp_i^n x||^{2\rho} + ||\wp_i^n y||^{2\rho}\right\} \right\} \end{cases}$$

$$= \begin{cases} \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty. \end{cases}$$

Thus, (3.34) holds. On the other hand

$$D(x,x) = K\left(\frac{x}{2}, \frac{x}{2}\right)$$

has the property $\frac{1}{\wp_i^{10}} D(\wp_i x, \wp_i x) = L\, D(x,x)$ for all $x \in \mathcal{E}$, where $K(x,x)$ is defined in (3.27). Hence,

$$D(x,x) = K\left(\frac{x}{2}, \frac{x}{2}\right)$$

(3.43)

$$= \begin{cases} \dfrac{1025\Theta}{3628800}, \\ \dfrac{\Theta||x||^\rho}{362880 \cdot 2^\rho}\left[2 \cdot 5^\rho + 20 \cdot 4^\rho + 90 \cdot 3^\rho + 241 \cdot 2^\rho + 1444\right] \\ \dfrac{\Theta||x||^{2\rho}}{1814400 \cdot 2^{2\rho}}\left[5^\rho + 10 \cdot 4^\rho + 45 \cdot 3^\rho + 120 \cdot 2^\rho + 210\right] \\ \dfrac{\Theta||x||^\rho}{2^\rho}\left\{\dfrac{1}{3628800}\left[2 \cdot 5^{2\rho} + 20 \cdot 4^{2\rho} + 90 \cdot 3^{2\rho} + 241 \cdot 2^{2\rho} + 1444\right]\right. \\ \left. + \dfrac{1}{1814400}\left[5^\rho + 10 \cdot 4^\rho + 45 \cdot 3^\rho + 120 \cdot 2^\rho + 210\right]\right\} \end{cases}$$

for all $x \in \mathcal{E}$. It follows from (3.43),

$$D(x,x) = K\left(\frac{x}{2}, \frac{x}{2}\right) = \begin{cases} \Theta_1, \\ \Theta_2, \\ \Theta_3, \\ \Theta_4 \end{cases} \tag{3.44}$$

where

$$\begin{aligned} \Theta_1 &= \frac{1025\Theta}{3628800}, \\ \Theta_2 &= \frac{\Theta||x||^\rho}{362880 \cdot 2^\rho}\left[2 \cdot 5^\rho + 20 \cdot 4^\rho + 90 \cdot 3^\rho + 241 \cdot 2^\rho + 1444\right] \\ \Theta_3 &= \frac{\Theta||x||^{2\rho}}{1814400 \cdot 2^{2\rho}}\left[5^\rho + 10 \cdot 4^\rho + 45 \cdot 3^\rho + 120 \cdot 2^\rho + 210\right] \end{aligned}$$

$$\Theta_4 = \frac{\Theta||x||^\rho}{2^\rho}\left\{\frac{1}{3628800}\Big[2\cdot 5^{2\rho}+20\cdot 4^{2\rho}+90\cdot 3^{2\rho}+241\cdot 2^{2\rho}+1444\Big] + \frac{1}{1814400}\Big[5^\rho+10\cdot 4^\rho+45\cdot 3^\rho+120\cdot 2^\rho+210\Big]\right\}$$

for all $x \in \mathcal{E}$. Similarly by (3.43), we prove

$$\frac{1}{\wp_i^{10}}D(\wp_i x, \wp_i x) = \begin{cases} \wp_i^{-10}\,\Theta_1, \\ \wp_i^{\rho-10}\Theta_2 \\ \wp_i^{2\rho-10}\Theta_3 \\ \wp_i^{2\rho-10}\Theta_4 \end{cases}$$

Hence, the inequality (3.38) holds for

(i) $L = \wp_i^{-10}$ if $i = 0$ and $L = \frac{1}{\wp_i^{-10}}$ if $i = 1$;
(ii) $L = \wp_i^{\rho-10}$ for $\rho < 10$ if $i = 0$ and $L = \frac{1}{\wp_i^{\rho-10}}$ for $\rho > 10$ if $i = 1$;
(iii) $L = \wp_i^{2\rho-10}$ for $2\rho > 10$ if $i = 0$ and $L = \frac{1}{\wp_i^{2\rho-10}}$ for $2\rho > 10$ if $i = 1$;
(iv) $L = \wp_i^{2\rho-10}$ for $2\rho > 10$ if $i = 0$ and $L = \frac{1}{\wp_i^{2\rho-10}}$ for $2\rho > 10$ if $i = 1$.

Now, from (3.38), we prove the following cases for condition (i).

$$\begin{array}{ll}
L = \wp_i^{-10}, i = 0 & L = \frac{1}{\wp_i^{-10}}, i = 1 \\
L = 2^{-10}, i = 0 & L = \frac{1}{2^{-10}}, i = 1 \\
L = 2^{-10}, i = 0 & L = 2^{10}, i = 1 \\
\parallel g(x) - \mathcal{T}(x) \parallel & \parallel g(x) - \mathcal{T}(x) \parallel \\
\quad \le \left(\frac{L^{1-i}}{1-L}\right) D(x,x) & \quad \le \left(\frac{L^{1-i}}{1-L}\right) D(x,x) \\
\quad = \left(\frac{(2^{-10})^{1-0}}{1-2^{-10}}\right)\Theta_1 & \quad = \left(\frac{(2^{10})^{1-1}}{1-2^{10}}\right)\Theta_1 \\
\quad = \left(\frac{2^{-10}}{1-2^{-10}}\right)\Theta_1 & \quad = \left(\frac{1}{1-2^{10}}\right)\Theta_1 \\
\quad = \left(\frac{\Theta_1}{2^{10}-1}\right) & \quad = \left(\frac{\Theta_1}{1-2^{10}}\right)
\end{array}$$

Also, from (3.38), we show the following cases for condition (ii).

$$\begin{array}{ll}
L = \wp_i^{\rho-10}, \rho < 10, i = 0 & L = \frac{1}{\wp_i^{\rho-10}}, \rho > 10, i = 1 \\
L = 2^{\rho-10}, \rho < 10, i = 0 & L = \frac{1}{2^{\rho-10}}, \rho < 10, i = 1 \\
L = 2^{\rho-10}, \rho < 10, i = 0 & L = 2^{10-\rho}, \rho > 10, i = 1 \\
\parallel g(x) - \mathcal{T}(x) \parallel & \parallel g(x) - \mathcal{T}(x) \parallel
\end{array}$$

$$\leq \left(\frac{L^{1-i}}{1-L}\right) D(x,x) \qquad\qquad \leq \left(\frac{L^{1-i}}{1-L}\right) D(x,x)$$
$$= \left(\frac{(2^{\rho-10})^{1-0}}{1-2^{\rho-10}}\right)\Theta_2 \qquad\qquad = \left(\frac{(2^{10-\rho})^{1-1}}{1-2^{10-\rho}}\right)\Theta_2$$
$$= \left(\frac{2^{\rho-10}}{1-2^{\rho-10}}\right)\Theta_2 \qquad\qquad = \left(\frac{1}{1-2^{10-\rho}}\right)\Theta_2$$
$$= \left(\frac{2^{\rho}}{2^{10}-2^{\rho}}\right)\Theta_2 \qquad\qquad = \left(\frac{2^{\rho}}{2^{\rho}-2^{10}}\right)\Theta_2$$

Once more, from (3.38), we have the following cases for condition (iii).

$$L = \wp_i^{2\rho-10}, 2\rho < 10, i = 0 \qquad\qquad L = \frac{1}{\wp_i^{2\rho-10}}, 2\rho > 10, i = 1$$
$$L = 2^{2\rho-10}, 2\rho < 10, i = 0 \qquad\qquad L = \frac{1}{2^{2\rho-10}}, 2\rho < 10, i = 1$$
$$L = 2^{2\rho-10}, 2\rho < 10, i = 0 \qquad\qquad L = 2^{10-2\rho}, 2\rho > 10, i = 1$$
$$\| g(x) - \mathcal{T}(x) \| \qquad\qquad \| g(x) - \mathcal{T}(x) \|$$
$$\leq \left(\frac{L^{1-i}}{1-L}\right) D(x,x) \qquad\qquad \leq \left(\frac{L^{1-i}}{1-L}\right) D(x,x)$$
$$= \left(\frac{(2^{2\rho-10})^{1-0}}{1-2^{2\rho-10}}\right)\Theta_3 \qquad\qquad = \left(\frac{(2^{10-2\rho})^{1-1}}{1-2^{10-2\rho}}\right)\Theta_3$$
$$= \left(\frac{2^{2\rho-10}}{1-2^{2\rho-10}}\right)\Theta_3 \qquad\qquad = \left(\frac{1}{1-2^{10-2\rho}}\right)\Theta_3$$
$$= \left(\frac{2^{2\rho}}{2^{10}-2^{2\rho}}\right)\Theta_3 \qquad\qquad = \left(\frac{2^{2\rho}}{2^{2\rho}-2^{10}}\right)\Theta_3$$

Finally, the proof of (3.38) for condition (iv) is similar to the condition (iii). □

4. Stability results in generalized 2-normed spaces

In this section, the generalized Ulam-Hyers stability of the decic functional equation (1.10) in generalized 2-normed space is discussed.

4.1. Definitions On generalized 2-normed space

We present some basic definitions related to generalized 2-normed spaces.

Definition 4.1. [1] Let X be linear space. A function $N(.,.) : X \times X \longrightarrow [0,\infty)$ is called a *generalized 2-normed* space if it satisfies the following conditions:

(G2N1) $N(x,y) = 0$ if and only if x and y are linearly independent vectors;
(G2N2) $N(x,y) = N(y,x)$ for all $x,y \in X$;
(G2N3) $N(\lambda x,y) = |\lambda| N(x,y)$ for all $x,y \in X$ and $X = \varphi, \varphi$ is a real or complex field;
(G2N4) $N(x+y,z) \leq N(x,z) + N(y,z)$ for all $x,y,z \in X$.

The generalized 2-normed space is denoted by $(X, N(.,.))$.

Definition 4.2. [1] A sequence $\{x_n\}$ in a generalized 2-normed space $(X, N(.,.))$ is called convergent if there exist $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, y) = 0$ then $\lim_{n\to\infty} N(x_n, y) = N(x, y)$ for all $y \in X$.

Definition 4.3. [1] A sequence $\{x_n\}$ in a generalized 2-normed space $(X, N(.,.))$ is called Cauchy sequence is there exist two lineary independent elements y and z in X such that $\{N(x_n, y)\}$ and $\{N(x_n, z)\}$ are real Cauchy sequences.

Definition 4.4. [1] A generalized 2-normed space $(X, N(.,.))$ is called generalized 2-Banach space is every Cauchy sequence is convergent.

In the section, we present the generalized Ulam-Hyers-Rassias stability for the functional equation (1.10).Throughout this section, let us consider $\mathcal{G}$ be a generalized 2-normed space and $\mathcal{H}$ be generalized 2-Banach space.

The proof of the subsequent theorems and corollaries are similar to the Theorems 3.1, 3.5 and Corollaries 3.2, 3.3, 3.6. Hence, the details of the proofs are omitted.

4.2. G2NS: Direct method

Theorem 4.5. *Let $b = \pm 1$ and $\kappa, K : \mathcal{G}^2 \longrightarrow [0, \infty)$ be a function such that*

$$\sum_{d=0}^{\infty} \frac{\kappa\left((2^{bd}x, u), (2^{bd}y, u)\right)}{2^{10bd}} \text{ converges in } \mathbb{R} \text{ and } \lim_{d\to\infty} \frac{\kappa\left((2^{bd}x, u), (2^{bd}y, u)\right)}{2^{10bd}} = 0$$

for all $x, y \in \mathcal{G}$ and all $u \in \mathcal{G}$. Let $Dg_{10} : \mathcal{G} \longrightarrow \mathcal{H}$ be a mapping fulfilling the inequality

$$N\left(Dg_{10}(x, y), u\right) \leq \kappa\left((x, u), (y, u)\right) \tag{4.1}$$

for all $x, y \in \mathcal{G}$ and all $u \in \mathcal{G}$. Then there exists a unique decic mapping $\mathcal{T} : \mathcal{G} \longrightarrow \mathcal{H}$ which satisfies (1.10) and

$$N\left(g(x) - \mathcal{T}(x), u\right) \leq \frac{1}{2^{10}} \sum_{d=\frac{1-b}{2}}^{\infty} \frac{K\left((2^{bd}x, u), (2^{bd}x, u)\right)}{2^{10bd}} \tag{4.2}$$

where $K\left((2^{bd}x, u), (2^{bd}x, u)\right)$ and $\mathcal{T}(x)$ are defined by

$$\begin{aligned}
&K\left((2^{bd}x,u),(2^{bd}x,u)\right)\\
&=\frac{1}{907200}\left\{\frac{1}{2}\left[\frac{1}{2}\cdot\kappa(0,(2\cdot 2^{bd}x,u))+\kappa((5\cdot 2^{bd}x,u),(2^{bd}x,u))\right.\right.\\
&\quad+10\kappa((4\cdot 2^{bd}x,u),(2^{bd}x,u))\\
&\quad\left.+45\kappa((3\cdot 2^{bd},u),(2^{bd},u))+120\kappa((2\cdot 2^{bd},u),(2^{bd},u))\right]\\
&\quad\left.+105\kappa((2^{bd},u),(2^{bd},u))+63\kappa(0,(2^{bd},u))\right\}
\end{aligned}$$

and

$$\lim_{d\to\infty} N\left(\mathcal{T}(x)-\lim_{d\to\infty}\frac{g(2^{bd}x)}{2^{10bd}},u\right)=0$$

for all $x\in\mathcal{G}$ and all $u\in\mathcal{G}$, respectively.

Corollary 4.6. *Let $Dg_{10}:\mathcal{G}\longrightarrow\mathcal{H}$ be a mapping. If there exist real numbers θ and λ such that*

$$\begin{aligned}
&N\left(Dg_{10}(x,y),u\right)\\
&\leq\begin{cases}\theta, & \\ \theta\left\{||(x,u)||^{\lambda}+||(y,u)||^{\lambda}\right\}, & \lambda\neq 10;\\ \theta||(x,u)||^{\lambda}||(y,u)||^{\lambda}, & \lambda\neq 5;\\ \theta\left\{||(x,u)||^{\lambda}||(y,u)||^{\lambda}+\left\{||(x,u)||^{2\lambda}+||(y,u)||^{2\lambda}\right\}\right\}, & \lambda\neq 5;\end{cases}
\end{aligned}$$

for all $x,y\in\mathcal{G}$ and all $u\in\mathcal{G}$, then there exists a unique decic mapping $\mathcal{T}:\mathcal{G}\longrightarrow\mathcal{H}$ such that

$$N\left(g(x)-\mathcal{T}(x),u\right)\leq\begin{cases}\dfrac{\theta_C}{|2^{10}-1|}, & \\ \dfrac{\theta_S||x,u||^{s}}{|2^{10}-2^{\lambda}|}, & \lambda\neq 10,\\ \dfrac{\theta_P||x,u||^{2\lambda}}{|2^{10}-2^{2\lambda}|}, & \lambda\neq 5,\\ \dfrac{\theta_{SP}||x,u||^{2\lambda}}{|2^{10}-2^{2\lambda}|}, & \lambda\neq 5,\end{cases}$$

where

$$\theta_C=\frac{1025\,\theta}{3628800},$$

$$\theta_S = \frac{\theta\Big[2 \cdot 5^{\lambda} + 20 \cdot 4^{\lambda} + 90 \cdot 3^{\lambda} + 241 \cdot 2^{\lambda} + 1444\Big]}{3628800},$$
$$\theta_P = \frac{\theta\Big[5^{\lambda} + 10 \cdot 4^{\lambda} + 45 \cdot 3^{\lambda} + 120 \cdot 2^{\lambda} + 210\Big]}{1814400},$$
$$\theta_{SP} = \frac{\theta}{3628800}\Big[2 \cdot 5^{2\lambda} + 20 \cdot 4^{2\lambda} + 90 \cdot 3^{2\lambda} + 241 \cdot 2^{2\lambda} + 1444\Big] + \frac{\theta}{1814400}\Big[5^{\lambda} + 10 \cdot 4^{\lambda} + 45 \cdot 3^{\lambda} + 120 \cdot 2^{\lambda} + 210\Big],$$

for all $x \in \mathcal{G}$ and all $u \in \mathcal{G}$.

Corollary 4.7. *Let $Dg_{10} : \mathcal{G} \longrightarrow \mathcal{H}$ be a mapping. If there exist real numbers $\theta, \lambda_1, \lambda_2, \lambda = \lambda_1 + \lambda_2$ such that*

$$N\left(Dg_{10}(x, y), u\right) \leq \begin{cases} \theta\left\{||(x, u)||^{\lambda_1} + ||(y, u)||^{\lambda_2}\right\}, & \lambda_1, \lambda_2 \neq 10; \\ \theta||(x, u)||^{\lambda_1}||(y, u)||^{\lambda_2}, & \lambda \neq 10; \\ \theta\left\{||(x, u)||^{\lambda_1}||y||^{\lambda_2} + \left\{||(x, u)||^{\lambda} + ||(y, u)||^{\lambda}\right\}\right\}, & \lambda \neq 10; \end{cases}$$

for all $x, y \in \mathcal{G}$ and all $u \in \mathcal{G}$, then there exists a unique decic mapping $\mathcal{T} : \mathcal{G} \longrightarrow \mathcal{H}$ such that

$$N\left(g(x) - \mathcal{T}(x), u\right) \leq \begin{cases} \theta_s, & \lambda_1, \lambda_2 \neq 10, \\ \theta_p, & \lambda \neq 10, \\ \theta_{sp}, & \lambda \neq 10, \end{cases}$$

where

$$\theta_s = \frac{1}{1814400}\left\{\frac{\theta||x, u||^{\lambda_1}\Big[5^{\lambda_1} + 10 \cdot 4^{\lambda_1} + 45 \cdot 3^{\lambda_1} + 120 \cdot 2^{\lambda_1} + 210\Big]}{|2^{10} - 2^{\lambda_1}|} + \frac{\theta||x||^{\lambda_2}\Big[2^{\lambda_2} + 688\Big]}{|2^{10} - 2^{\lambda_2}|}\right\},$$
$$\theta_p = \frac{\theta||x, u||^{\lambda_1+\lambda_2}}{1814400}\left\{\frac{\Big[5^{\lambda_1} + 10 \cdot 4^{\lambda_1} + 45 \cdot 3^{\lambda_1} + 120 \cdot 2^{\lambda_1} + 210\Big]}{|2^{10} - 2^{\lambda_1+\lambda_2}|}\right\},$$
$$\theta_{sp} = \frac{\theta||x, u||^{\lambda}}{1814400\ |2^{10} - 2^{\lambda_1+\lambda_2}|}\left\{\Big[5^{\lambda} + 10 \cdot 4^{\lambda} + 45 \cdot 3^{\lambda} + 120 \cdot 2^{\lambda} + 210\Big] + \Big[2^{\lambda} + 688\Big] + \Big[5^{\lambda_1} + 10 \cdot 4^{\lambda_1} + 45 \cdot 3^{\lambda_1} + 120 \cdot 2^{\lambda_1} + 210\Big]\right\},$$

for all $x \in \mathcal{G}$ and all $u \in \mathcal{G}$

4.3. G2NS: Fixed point method

THEOREM 4.8. *Let $Dg_{10} : \mathcal{G} \longrightarrow \mathcal{H}$ be a mapping for which there exists a function $\zeta : \mathcal{G}^2 \longrightarrow [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\wp_i^{10n}} \zeta((\wp_i^n x, u), (\wp_i^n y, u)) = 0$$

where

$$\wp_i = \begin{cases} 2 & if \quad i = 0, \\ \frac{1}{2} & if \quad i = 1 \end{cases}$$

such that the functional inequality

$$N\left(Dg_{10}(x, y), u\right) \leq \zeta((x, u), (y, u))$$

holds for all $x, y \in \mathcal{G}$ and for all $u \in \mathcal{G}$. Assume that there exists $L = L(i)$ such that the mapping

$$D((x, u), (x, u)) = K\left(\left(\frac{x}{2}, u\right), \left(\frac{x}{2}, u\right)\right)$$

where $K((x, u), (x, u))$ is defined in (3.27) with the property

$$\frac{1}{\wp_i^{10}} D((\wp_i x, u), (\wp_i x, u)) = L\ D((x, u), (x, u))$$

for all $x \in \mathcal{G}$ and for all $u \in \mathcal{G}$. Then, there exists a unique decic mapping $\mathcal{T} : \mathcal{G} \longrightarrow \mathcal{H}$ satisfying the functional equation (1.10) and

$$N\left(g(x) - \mathcal{T}(x), u\right) \leq \left(\frac{L^{1-i}}{1 - L}\right) D((x, u), (x, u))$$

for all $x \in \mathcal{G}$ and for all $u \in \mathcal{G}$.

COROLLARY 4.9. *Let $Dg_{10} : \mathcal{G} \longrightarrow \mathcal{H}$ be a mapping. If there exist real numbers Θ and ρ such that*

$$\|Dg_{10}(x, y)\| \leq \begin{cases} \Theta, & \\ \Theta\left\{||x, u||^\rho + ||y, u||^\rho\right\} & \rho \neq 10; \\ \Theta||x, u||^\rho ||y, u||^\rho & \rho \neq 5; \\ \Theta\left\{||x, u||^\rho ||y, u||^\rho + \left\{||x, u||^{2\rho} + ||y, u||^{2\rho}\right\}\right\} & \rho \neq 5; \end{cases}$$

for all $x, y \in \mathcal{G}$ and for all $u \in \mathcal{G}$, then there exists a unique decic mapping $\mathcal{T} : \mathcal{G} \longrightarrow \mathcal{H}$ such that

$$\|g(x) - \mathcal{T}(x)\| \leq \begin{cases} \dfrac{\Theta_1}{|2^{10} - 1|}, & \\ \dfrac{\Theta_2}{|2^{10} - 2^{\rho}|}, & \rho \neq 10, \\ \dfrac{\Theta_3}{|2^{10} - 2^{2\rho}|}, & \rho \neq 5, \\ \dfrac{\Theta_4}{|2^{10} - 2^{2\rho}|}, & \rho \neq 5, \end{cases}$$

where

$$\begin{aligned} \Theta_1 &= \frac{1025\Theta}{3628800}, \\ \Theta_2 &= \frac{\Theta||x||^{\rho}}{362880 \cdot 2^{\rho}} \Big[2 \cdot 5^{\rho} + 20 \cdot 4^{\rho} + 90 \cdot 3^{\rho} + 241 \cdot 2^{\rho} + 1444\Big] \\ \Theta_3 &= \frac{\Theta||x||^{2\rho}}{1814400 \cdot 2^{2\rho}} \Big[5^{\rho} + 10 \cdot 4^{\rho} + 45 \cdot 3^{\rho} + 120 \cdot 2^{\rho} + 210\Big] \\ \Theta_4 &= \frac{\Theta||x||^{\rho}}{2^{\rho}} \left\{ \frac{1}{3628800} \Big[2 \cdot 5^{2\rho} + 20 \cdot 4^{2\rho} + 90 \cdot 3^{2\rho} + 241 \cdot 2^{2\rho} + 1444\Big] \right. \\ &\qquad \left. + \frac{1}{1814400} \Big[5^{\rho} + 10 \cdot 4^{\rho} + 45 \cdot 3^{\rho} + 120 \cdot 2^{\rho} + 210\Big] \right\} \end{aligned}$$

for all $x \in \mathcal{G}$ and for all $u \in \mathcal{G}$.

5. Stability results in random normed spaces

In this section, the generalized Ulam-Hyers stability of the decic functional equation (1.10) in RN-space is provided.

5.1. Definitions and notations

In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces as in [61, 62].

From now on, Δ^+ is the space of distribution functions, that is, the space of all mappings

$$F : R \cup \{-\infty, \infty\} \longrightarrow [0, 1],$$

such that F is leftcontinuous and nondecreasing on $R, F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$

for which $l^{-}F(+\infty) = 1$, where $l^{-}f(x)$ denotes the left limit of the function f at the point x, that is,

$$l^{-}f(x) = \lim_{t \to x^{-}} f(t).$$

The space Δ^{+} is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^{+} in this order is the distribution function ϵ_0 given by

$$\epsilon_0(t) = \begin{cases} 0, & if\ t \leq 0, \\ 1, & if\ t \geq 0. \end{cases} \tag{5.1}$$

Definition 5.1. [61] A mapping $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a continuous triangular norm (briefly, a continuous $t-$norm) if T satisfies the following conditions:

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a, 1) = a$ for all $a \in [0,1]$;
(d) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Typical examples of continuous $t-$norms are $T_P(a, b) = ab, T_M(a, b) = min(a, b)$ and $T_L(a, b) = max(a + b - 1, 0)$ (the Lukasiewicz $t-$norm). Recall (see [29, 30]) that if T is a $t-$norm and x_n is a given sequence of numbers in $[0,1]$, then $T_{i=1}^{n} x_{n+i}$ is defined recurrently by

$$T_{i=1}^{1} x_i = x_1 \ and\ T_{i=1}^{n} x_i = T\left(T_{i=1}^{n-1} x_i, x_n\right)\ for\ n \geq 2.$$

$T_{i=n}^{\infty} x_i$ is defined as $T_{i=1}^{\infty} x_{n+i}$. It is known [30] that, for the Lukasiewicz $t-$norm, the following implication holds:

$$\lim_{n \to \infty} (T_L)_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty} (1 - x_n) < \infty \tag{5.2}$$

Definition 5.2. [62] A random normed space (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous $t-$norm and μ is a mapping from X into D^{+} satisfying the following conditions:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(RN2) $\mu_{\alpha\, x}(t) = \mu_x(t/|\alpha|)$ for all $x \in X$, and $\alpha \in \mathbb{R}$ with $\alpha \neq 0$;
(RN3) $\mu_{x+y}(t + s) \geq T\left(\mu_x(t), \mu_y(s)\right)$ for all $x, y \in X$ and $t, s \geq 0$.

Example 5.3. *Every normed spaces* $(X, ||\cdot||)$ *defines a random normed space* (X, μ, T_M), *where*

$$\mu_x(t) = \frac{t}{t + ||x||}$$

and T_M is the minimum $t-$norm. This space is called the induced random normed space.

Definition 5.4. Let (X, μ, T) be a RN-space.

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\varepsilon) > 1 - \lambda$ for all $n \geq N$.
(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for any $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\varepsilon) > 1 - \lambda$ for all $n \geq m \geq N$.
(3) A RN-space (X, μ, T) is said to be complete if every Cauchy sequence in X is convergent to a point in X.

Theorem 5.5. [62] *If (X, μ, T) is a RN-space and $\{x_n\}$ is a sequence in X such that $x_n \to x$, then $\lim\limits_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Hereafter through out this section, let us consider X be a linear space and (Y, μ, T) is a complete RN-space.

5.2. RN Space: Direct method

Theorem 5.6. *Let $j = \pm 1$. Let $Dg_{10} : X \longrightarrow Y$ be a mapping for which there exist a function $\eta : X^2 \longrightarrow D^+$ with the condition*

$$T_{i=0}^{\infty}\left(\mu_{\eta(2^{ij}x, 2^{ij}x)}\left(2^{10ij}\ s\right)\right) = 1 = \lim_{n\to\infty} \mu_{\eta(2^{nj}x, 2^{nj}y)}\left(2^{10nj}\ s\right) \tag{5.3}$$

such that the functional inequality such that

$$\mu_{Dg_{10}(x,y)}(s) \geq \eta_{x,y}(s) \tag{5.4}$$

for all $x, y \in X$ and all $s > 0$. Then, there exists a unique decic mapping $\mathcal{T} : X \longrightarrow Y$ satisfying the functional equation (1.10) *and*

$$\mu_{\mathcal{T}(x) - g(x)}\left(\frac{1025\ s}{3628800}\right) \geq T_{i=0}^{\infty}\left(E_{2^{ij}x, 2^{ij}x}\left(2^{10(i+1)j}\ s\right)\right) \tag{5.5}$$

where $E_{x,x}(s)$ and $\mathcal{T}(x)$ are defined by

$$E_{x,x}(s) = T^6\Big(\eta_{0,2x}(s), \eta_{5x,x}(s), \eta_{4x,x}(s), \eta_{3x,x}(s), \eta_{2x,x}(s), \eta_{x,x}(s), \eta_{0,x}(s)\Big) \tag{5.6}$$

and

$$\mu_{\mathcal{T}(x)}(s) = \lim_{n\to\infty} \mu_{\frac{g(2^{nj}x)}{2^{10nj}}}(s) \tag{5.7}$$

for all $x \in X$ and all $s > 0$.

Proof. We bring the proof for $j = 1$. For the case of $j = -1$ is similar. Replacing (x, y) by $(0, 2x)$ in (5.4), we get

$$\mu_{2g(10x)-20g(8x)+90g(6x)-240g(4x)-3628380g(2x)}(s) \geq \eta_{0,2x}(s) \tag{5.8}$$

for all $x \in X$ and all $s > 0$. Using (RN2) in (5.8), we obtain

$$\mu_{g(10x)-10g(8x)+45g(6x)-120g(4x)-1814190g(2x)}\left(\frac{s}{2}\right) \geq \eta_{0,2x}(s) \tag{5.9}$$

for all $x \in X$ and all $s > 0$. Again setting (x, y) by $(5x, x)$ in (5.4), we have

$$\begin{aligned}&\mu_{g(10x)-10g(9x)+45g(8x)-120g(7x)+210g(6x)-252g(5x)}(s)\\&{}_{+210g(4x)-120g(3x)+45g(2x)-3628810g(x)} \geq \eta_{5x,x}(s)\end{aligned} \tag{5.10}$$

for all $x \in X$ and all $s > 0$. Combining (5.9) and (5.10), we arrive at

(5.11)

$$\begin{aligned}
&\mu_{10g(9x)-55g(8x)+120g(7x)-165g(6x)+252g(5x)-330g(4x)+120g(3x)}\\
&{}_{-1814235g(2x)+3628810g(x)}\left(\frac{3}{2}s\right)\\
&= \mu_{g(10x)-10g(8x)+45g(6x)-120g(4x)-1814190g(2x)-g(10x)+10g(9x)-45g(8x)+120g(7x)}\\
&{}_{-210g(6x)+252g(5x)-210g(4x)+120g(3x)-45g(2x)+3628810g(x)}\left(\frac{3}{2}s\right)\\
&\geq T\Big(\mu_{g(10x)-10g(8x)+45g(6x)-120g(4x)-1814190g(2x)}\left(\frac{s}{2}\right),\\
&\mu_{g(10x)-10g(9x)+45g(8x)-120g(7x)+210g(6x)-252g(5x)+210g(4x)-120g(3x)}g\\
&{}_{+45g(2x)-3628810g(x)}(s)\Big)\\
&\geq T\Big(\eta_{0,2x}(s), \eta_{5x,x}(s)\Big)
\end{aligned}$$

for all $x \in X$ and all $s > 0$. Substituting (x, y) by $(4x, x)$ in (5.4) and using evenness of g, we get

$$\begin{aligned}&\mu_{g(9x)-10g(8x)+45g(7x)-120g(6x)+210g(5x)-252g(4x)+210g(3x)}\\&{}_{-120g(2x)-3628754g(x)}(s) \geq \eta_{4x,x}(s)\end{aligned} \tag{5.12}$$

for all $x \in X$ and all $s > 0$. Using (RN2) in (5.12), one obtains

$$\begin{aligned}&\mu_{10g(9x)-100g(8x)+450g(7x)-1200g(6x)+2100g(5x)-2520g(4x)}\\&{}_{+2100g(3x)-1200g(2x)-36287540g(x)}(10s) \geq \eta_{4x,x}(s)\end{aligned} \tag{5.13}$$

for all $x \in X$ and all $s > 0$. Combining (5.12) and (5.12), we find

$$\mu_{45g(8x)-330g(7x)+1035g(6x)-1848g(5x)+2190g(4x)-1980g(3x) -1813035g(2x)+39916350g(x)}\left(\frac{23}{2}s\right) \geq T^2\Big(\eta_{0,2x}(s), \eta_{5x,x}(s), \eta_{4x,x}(s)\Big) \tag{5.14}$$

for all $x \in X$ and all $s > 0$. Letting (x, y) by $(3x, x)$ in (5.4) and using evenness of g, we have

$$\mu_{g(8x)-10g(7x)+45g(6x)-120g(5x)+210g(4x)-252g(3x) +211g(2x)-3628930g(x)}(s) \geq \eta_{3x,x}(s) \tag{5.15}$$

for all $x \in X$ and all $s > 0$. Using (RN2) in (5.15), we get

$$\mu_{45g(8x)-450g(7x)+2025g(6x)-5400g(5x)+9450g(4x)-11340g(3x) +9495g(2x)-163301850g(x)}(45s) \geq \eta_{3x,x}(s) \tag{5.16}$$

for all $x \in X$ and all $s > 0$. Plugging (5.15) into (5.16), we arrive at

$$\mu_{120(7x)-990g(6x)+3552g(5x)-7260(4x)+9360g(3x) -1822530g(2x)+203218200g(x)}\left(\frac{113}{2}s\right) \geq T^3\Big(\eta_{0,2x}(s), \eta_{5x,x}(s), \eta_{4x,x}(s), \eta_{3x,x}(s)\Big) \tag{5.17}$$

for all $x \in X$ and all $s > 0$. Switching (x, y) into $(2x, x)$ in (5.7) and using evenness of g, we obtain

$$\mu_{g(7x)-10g(6x)+45g(5x)-120g(4x)+211g(3x)-262g(2x)-3628545g(x)}(s) \geq \eta_{2x,x}(s)) \tag{5.18}$$

for all $x \in X$ and all $s > 0$. Using (RN2) in (5.18), one can show that

$$\mu_{120g(7x)-1200g(6x)+5400g(5x)-14400g(4x)+25320g(3x) -31440g(2x)-435425400g(x)}(120s) \geq \eta_{2x,x}(s) \tag{5.19}$$

for all $x \in X$ and all $s > 0$. Combining (5.17) and (5.19), we reach

$$\mu_{210g(6x)-1848g(5x)+7140g(4x)-15960g(3x)-1791090g(2x) +638643600g(x)}\left(\frac{353}{2}s\right) \geq T^4\Big(\eta_{0,2x}(s), \eta_{5x,x}(s), \eta_{4x,x}(s), \eta_{3x,x}(s), \eta_{2x,x}(s)\Big) \tag{5.20}$$

for all $x \in X$ and all $s > 0$. Interchanging (x, y) into (x, x) in (5.7) and using evenness of g, we get

$$\mu_{g(6x)-10g(5x)+46g(4x)-130g(3x)+255g(2x)-3629172g(x)}(s) \geq \eta_{x,x}(s) \tag{5.21}$$

for all $x \in X$ and all $s > 0$. It follows from (5.21) that

$$\begin{aligned}&\mu_{210g(6x)-2100g(5x)+9660g(4x)-27300g(3x)} \\ &\quad {}_{+53550g(2x)-762126120g(x)}(120s) \geq \eta_{x,x}(s)\end{aligned} \tag{5.22}$$

for all $x \in X$ and all $s > 0$. Plugging (5.20) into (5.22), we have

$$\begin{aligned}&\mu_{126g(5x)-1260g(4x)+5670g(3x)-922320g(2x)+700384860g(x)}\left(\frac{773}{2}s\right) \\ &\geq T^5\Big(\eta_{0,2x}(s), \eta_{5x,x}(s), \eta_{4x,x}(s), \eta_{3x,x}(s), \eta_{2x,x}(s), \eta_{x,x}(s)\Big)\end{aligned} \tag{5.23}$$

for all $x \in X$ and all $s > 0$. Replacing (x, y) by $(0, x)$ in (5.7), we obtain

$$\mu_{2g(5x)-20g(4x)+90g(3x)-240g(2x)-3628380g(x)}(s) \geq \eta_{0,x}(s) \tag{5.24}$$

for all $x \in X$ and all $s > 0$. Using (RN2) in (5.24) that

$$\mu_{126g(5x)-1260g(4x)+5670g(3x)-15120g(2x)-228587940g(x)}(126s) \geq \eta_{0,x}(s) \tag{5.25}$$

for all $x \in X$ and all $s > 0$. Combining (5.23) and (5.25), we find

$$\begin{aligned}&\mu_{-1814400g(2x)+1857945600g(x)}\left(\frac{1025}{2}s\right) \\ &\geq T^6\Big(\eta_{0,2x}(s), \eta_{5x,x}(s), \eta_{4x,x}(s), \eta_{3x,x}(s), \eta_{2x,x}(s), \eta_{x,x}(s), \eta_{0,x}(s)\Big)\end{aligned} \tag{5.26}$$

for all $x \in X$ and all $s > 0$. It follows from (5.26) that

$$\mu_{g(2x)-1024g(x)}\left(\frac{1}{1814400}\left[\frac{1025}{2}s\right]\right) \geq E_{x,x}(s) \tag{5.27}$$

for all $x \in X$ and all $s > 0$, where

$$E_{x,x}(s) = T^6\Big(\eta_{0,2x}(s), \eta_{5x,x}(s), \eta_{4x,x}(s), \eta_{3x,x}(s), \eta_{2x,x}(s), \eta_{x,x}(s), \eta_{0,x}(s)\Big)$$

for all $x \in X$ and all $s > 0$. The above equation can be rewritten as

$$\mu_{g(2x)-2^{10}g(x)}\left(\frac{1025s}{1814400 \cdot 2}\right) \geq E_{x,x}(s) \tag{5.28}$$

for all $x \in X$ and all $s > 0$. It follows from (5.28) and (RN2), we have

$$\mu_{\frac{g(2x)}{2^{10}}-g(x)}\left(\frac{1025s}{3628800 \cdot 2^{10}}\right) \geq E_{x,x}(s) \tag{5.29}$$

for all $x \in X$ and all $s > 0$. Replacing x by $2^n x$ in (5.29), we arrive

$$\mu_{\frac{g(2^{n+1}x)}{2^{10}} - g(2^n x)}\left(\frac{1025s}{3628800 \cdot 2^{10}}\right) \geq E_{2^n x, 2^n x}(s) \tag{5.30}$$

for all $x \in X$ and all $s > 0$. Changing s by $2^{10}s$ in (5.30), we reach

$$\mu_{\frac{g(2^{n+1}x)}{2^{10}} - g(2^n x)}\left(\frac{2^{10} \times 1025s}{3628800}\right) \geq E_{2^n x, 2^n x}(2^{10}s) \tag{5.31}$$

for all $x \in X$ and all $s > 0$. Using (RN2) in the above equation, we achieve

$$\mu_{\frac{g(2^{n+1}x)}{2^{10(n+1)}} - \frac{g(2^n x)}{2^{10n}}}\left(\frac{1025s}{3628800}\right) \geq E_{2^n x, 2^n x}(2^{10(n+1)}s) \tag{5.32}$$

for all $x \in X$ and all $s > 0$. It is easy to see that

$$\frac{g(2^n x)}{2^{10n}} - g(x) = \sum_{i=0}^{n-1} \frac{g(2^{i+1}x)}{2^{10(i+1)}} - \frac{g(2^i x)}{2^{10i}} \tag{5.33}$$

for all $x \in X$. From equations (5.32) and (5.33), we get

$$\begin{aligned}
\mu_{\frac{g(2^n x)}{2^{10n}} - g(x)}\left(\frac{1025\ s}{3628800}\right) &= \mu_{\sum_{i=0}^{n-1} \frac{g(2^{i+1}x)}{2^{10(i+1)}} - \frac{g(2^i x)}{2^{10i}}}\left(\frac{1025s}{3628800}\right) \\
&\geq T_{i=0}^{n-1} \mu_{\frac{g(2^{i+1}x)}{2^{10(i+1)}} - \frac{g(2^i x)}{2^{10i}}}\left(\frac{1025s}{3628800}\right) \\
&\geq T_{i=0}^{n-1} E_{2^i x, 2^i x}(2^{10(i+1)}s) \qquad (5.34)
\end{aligned}$$

for all $x \in X$ and all $s > 0$. Here, we show that the sequence $\left\{\frac{g(2^n x)}{2^{10n}}\right\}$ is convergent. We replace x by $2^m x$ in (5.34) and apply (RN2). Then

$$\begin{aligned}
\mu_{\frac{g(2^{n+m}x)}{2^{10(n+m)}} - \frac{g(2^m x)}{2^{10m}}}\left(\frac{1025s}{3628800}\right) &\geq T_{i=0}^{n-1} E_{2^{m+i}x, 2^{m+i}x}(2^{10(i+m+1)}s) \\
&= T_{i=m}^{m+n-1} E_{2^m x, 2^m x}(2^{10(i+1)}s) \qquad (5.35) \\
&\rightarrow 1 \ as\ m \rightarrow \infty
\end{aligned}$$

for all $x \in X$ and all $s > 0$. Thus, $\left\{\dfrac{g(2^n x)}{2^{10n}}\right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $\mathcal{T} : X \to Y$, we define

$$\mu_{\mathcal{T}(x)}(s) = \lim_{n\to\infty} \mu_{\frac{g(2^n x)}{2^{10n}}}(s)$$

for all $x \in X$ and all $s > 0$. Letting $m = 0$ and $n \to \infty$ in (5.35), we arrive (5.5) for all $x \in X$ and all $s > 0$. Now, we have to show that $\mathcal{T}$

satisfies (1.10). Replacing (x, y) by $(2^n x, 2^n y)$, we have

$$\mu_{\frac{1}{2^n} Dg_{10}(2^n x, 2^n y)}(s) \geq \eta_{2^n x, 2^n y}(2^n\ s) \tag{5.36}$$

for all $x \in X$ and all $s > 0$. Taking $n \to \infty$ both sides, we find that $\mathcal{T}$ satisfies (1.10) for all $x, y \in X$. Therefore the mapping $\mathcal{T} : X \to Y$ is decic. Lastly, to prove the uniqueness of the decic mapping $\mathcal{T}$ subject to (5.7), let us assume that exists an decic mapping $\mathcal{T}'$ which satisfies (5.5) and (5.7). Since $\mathcal{T}(2^n x) = 2^{10n}\mathcal{T}(x)$ and $\mathcal{T}'(2^n x) = 2^{10n}\mathcal{T}'(x)$ for all $x \in X$ and all $n \in \mathbb{N}$, it follows from (5.7) that

$$\begin{aligned}
&\mu_{\mathcal{T}(x)-\mathcal{T}'(x)}(2s) = \mu_{\mathcal{T}(2^n x)-\mathcal{T}'(2^n x)}(2^{10n+1} s)\\
&= \mu_{\mathcal{T}(2^n x)-g(2^n x)+g(2^n x)-\mathcal{T}'(2^n x)}(2^{10n+1} s)\\
&\geq T\left(\mu_{\mathcal{T}(2^n x)-g(2^n x)}(2^{10n} s), \mu_{g(2^n x)-\mathcal{T}'(2^n x)}(2^{10n} s)\right)\\
&\geq T\left(T_{i=0}^{\infty}\left(E_{2^{i+n}x, 2^{i+n}x}\right)(2^{10(i+n+1)} s), T_{i=0}^{\infty}\left(E_{2^{i+n}x, 2^{i+n}x}\right)(2^{10(i+n+1)} s)\right)\\
&\to\ 1\ as\ n\ \to\ \infty
\end{aligned}$$

for all $x \in X$ and all $s > 0$. Hence, $\mathcal{T}$ is unique. $\square$

The following Corollary is an immediate consequence of Theorem 5.6 concerning the stability of (1.10).

Corollary 5.7. *Let p and q be nonnegative real numbers. Let a mapping $g : X \longrightarrow Y$ satisfies the inequality*

$$\mu_{Dg_{10}(x,y)}(s) \geq \begin{cases} \eta_p(s) & \\ \eta_{p(||x||^q+||y||^q)}(s) & ; q \neq 10;\\ \eta_{p(||x||^q||y||^q)}(s) & ; q \neq 5;\\ \eta_{p(||x||^q||y||^q+[||x||^{2q}+||y||^{2q})}(s) & ; q \neq 5; \end{cases}$$

for all $x, y \in X$ and all $s > 0$. Then there exists a unique decic mapping $\mathcal{T} : X \longrightarrow Y$ satisfying the functional equation (1.10) and

$$\mu_{\mathcal{T}(x)-g(x)}(s) \geq \begin{cases} \eta_{p_1}(|2^{10}-1|s) & \\ \eta_{p_2}(|2^{10}-2^q|s), & q \neq 10,\\ \eta_{p_3}(|2^{10}-2^{2q}|s), & q \neq 5,\\ \eta_{p_4}(|2^{10}-2^{2q}|s), & q \neq 5, \end{cases}$$

where

$$p_1 = \frac{1025\ p}{3628800},$$

$$p_2 = \frac{p\Big[2 \cdot 5^q + 20 \cdot 4^q + 90 \cdot 3^q + 241 \cdot 2^q + 1444\Big]}{3628800},$$
$$p_3 = \frac{p\Big[5^q + 10 \cdot 4^q + 45 \cdot 3^q + 120 \cdot 2^q + 210\Big]}{1814400},$$
$$p_4 = \frac{p}{3628800}\Big[2 \cdot 5^{2q} + 20 \cdot 4^{2q} + 90 \cdot 3^{2q} + 241 \cdot 2^{2q} + 1444\Big] + \frac{p}{1814400}\Big[5^q + 10 \cdot 4^q + 45 \cdot 3^q + 120 \cdot 2^q + 210\Big],$$

for all $x \in X$ and all $s > 0$.

5.3. RNspace : Fixed point method

THEOREM 5.8. *Let $g : X \to Y$ be a mapping for which there exists a function $\eta : X^2 \longrightarrow D^+$ with the condition*

$$\lim_{n\to\infty} \mu_{\eta(\nu_i^n x, \nu_i^n y)}\left(\nu_i^n\ s\right) = 1 \tag{5.37}$$

where

$$\nu_i = \begin{cases} 2 & if \quad i = 0, \\ \frac{1}{2} & if \quad i = 1 \end{cases} \tag{5.38}$$

such that the functional inequality

$$\mu_{Dg_{10}(x,y)}(s) \geq \eta_{x,y}(s) \tag{5.39}$$

holds for all $x, y \in X$ and all $s > 0$. If there exists $L = L(i)$ such that the mapping

$$F_{x,x}(s) = E_{\frac{x}{2},\frac{x}{2}}(s),$$

has the property

$$F_{x,x}(L\ \nu_i\ s) = F_{\frac{x}{\nu_i},\frac{x}{\nu_i}}(s) \tag{5.40}$$

for all $x \in X$ and all $s > 0$, then there exists a unique decic mapping $\mathcal{T} : X \longrightarrow Y$ satisfying the functional equation (1.10) *and*

$$\mu_{\mathcal{T}(x)-g(x)}\left(\frac{L^{1-i}}{1-L}\ s\right) \geq F_{x,x}(s) \tag{5.41}$$

holds for all $x \in X$ and all $s > 0$.

Proof. Consider the set $S = \{g|\, g : X \longrightarrow Y,\ g(0) = 0\}$ and introduce the generalized metric $d : S \times S \longrightarrow [0, \infty]$ as follows:

$$d(g, h) = \inf\{K \in (0, \infty) : \mu_{g(x)-h(x)}(Ks) \geq F_{x,x}(s), x \in X, s > 0\}.$$

It is easy to see that (S, d) is complete with respect to the above metric. Now, define $J : S \to S$ by $Jg(x) = \frac{1}{\nu_i} g(\nu_i x)$, for all $x \in X$. for any $g, h \in S$, we have

$$\mu_{g(x)-h(x)}(Ks) \geq F_{x,x}(s), \qquad (x \in X, s > 0).$$

Indeed,

$$\mu_{\frac{1}{\nu_i} g(\nu_i x) - \frac{1}{\nu_i} h(\nu_i x)}(K\ \nu_i\ s) \geq F_{\nu_i x, \nu_i x}(s), \qquad (x \in X, s > 0).$$

The last inequality show that

$$\mu_{\frac{1}{\nu_i} g(\nu_i x) - \frac{1}{\nu_i} h(\nu_i x)}(K\ L\nu_i\ s) \geq F_{x,x}(s), \qquad (x \in X, s > 0).$$

Therefore, $\mu_{Jg(x)-Jh(x)}(K\ L\nu_i\ s) \geq F_{x,x}(s)$ for all $x \in X$ and $s > 0$. This implies $d(Jg, Jh) \leq L\nu_i d(g, h)$, for all $g, h \in S$. In other words, J is a strictly contractive mapping on S with Lipschitz constant $L = \nu_i$. The rest of the proof is similar to that of Theorem 3.5. $\square$

From Theorem 5.8, we obtain the following corollary concerning the generalized Ulam-Hyers stability for the functional equation (1.10).

Corollary 5.9. *Let p and q be nonnegative real numbers. Suppose that the mapping $g : X \longrightarrow Y$ satisfies the inequality*

$$\mu_{Dg_{10}(x,y)}(s) \geq \begin{cases} \eta_p(s) & \\ \eta_{p(||x||^q+||y||^q)}(s), & q \neq 10; \\ \eta_{p(||x||^q||y||^q)}(s), & q \neq 5; \\ \eta_{p(||x||^q||y||^q+[||x||^{2q}+||y||^{2q})}(s), & q \neq 5; \end{cases}$$

for all $x, y \in X$ and all $s > 0$. Then, there exists a unique decic mapping $\mathcal{T} : X \longrightarrow Y$ satisfying the functional equation (1.10) *and*

$$\mu_{\mathcal{T}(x)-g(x)}(s) \geq \begin{cases} \eta_{p_1}(|2^{10}-1|s) & \\ \eta_{p_2}(|2^{10}-2^q|s), & q \neq 10, \\ \eta_{p_3}(|2^{10}-2^{2q}|s), & q \neq 5, \\ \eta_{p_4}(|2^{10}-2^{2q}|s),, & q \neq 5, \end{cases}$$

where

$$p_1 = \frac{1025\ p}{3628800},$$
$$p_2 = \frac{p\Big[2 \cdot 5^q + 20 \cdot 4^q + 90 \cdot 3^q + 241 \cdot 2^q + 1444\Big]}{3628800},$$
$$p_3 = \frac{p\Big[5^q + 10 \cdot 4^q + 45 \cdot 3^q + 120 \cdot 2^q + 210\Big]}{1814400},$$
$$p_4 = \frac{p}{3628800}\Big[2 \cdot 5^{2q} + 20 \cdot 4^{2q} + 90 \cdot 3^{2q} + 241 \cdot 2^{2q} + 1444\Big] + \frac{p}{1814400}\Big[5^q + 10 \cdot 4^q + 45 \cdot 3^q + 120 \cdot 2^q + 210\Big],$$

for all $x \in X$ *and all* $s > 0$.

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